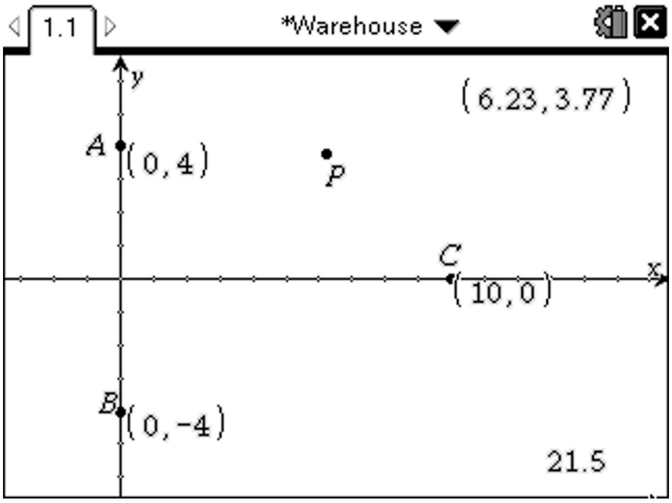
The Warehouse Problem

This activity will initially take you through the construction that is required to investigate a problem that is modelled on a real life situation.

You will be required to document your results and discoveries by writing them out, on paper.

We are going to create a construction on the TI-Nspire that looks like this:



The construction shows three stores being served by a warehouse. The three stores are located at A, B and C. The warehouse is located at the point P.

We are aiming to find out where to locate the warehouse, P, so that the delivery trucks travel as short a distance as possible.

We will assume that it is possible to drive directly to each store from the warehouse and that each store sells the same goods in the same amounts.

Let PA be the straight-line distance from the warehouse, P, to store A. Similarly PB is the straight-line distance from P to B. And same for PC.

In the screen shot, the total distance PA+PB+PC is equal to 21.5

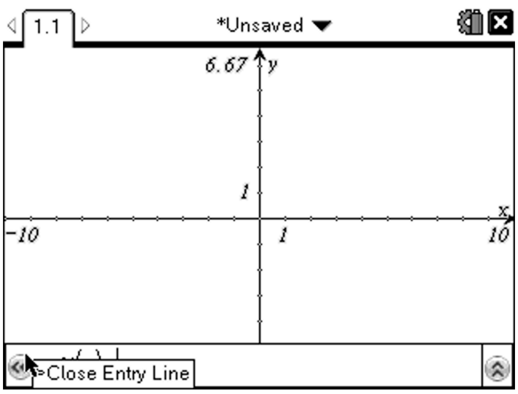
CONSTRUCTION STEPS

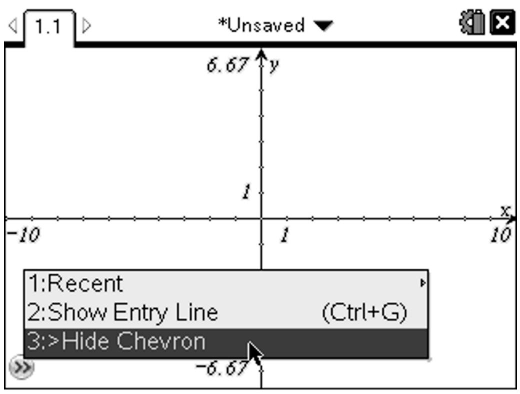
1. Start a new document with a Graphs application.

2. Save it under the name 'Warehouse'

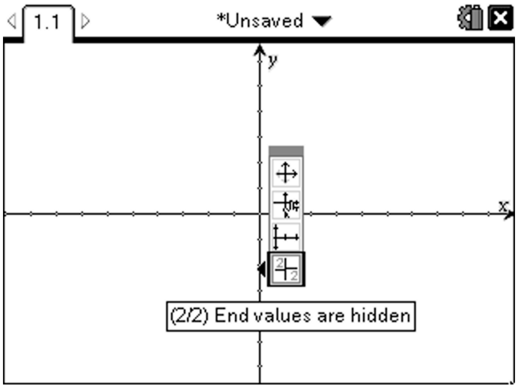
3. Click the chevron in the bottom left corner, to Close the Entry Line *[see screen shot, below left]*

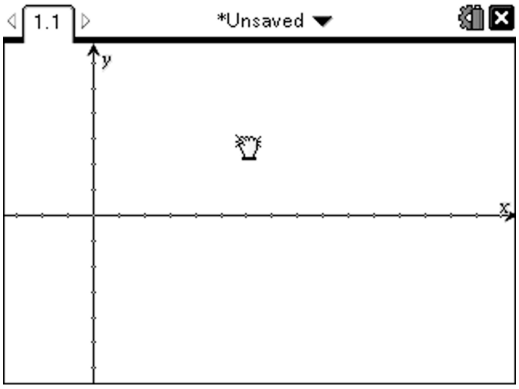
4. (ctrl) and (menu) the chevron again, so that you can hide it *[see screen shot, below right]*



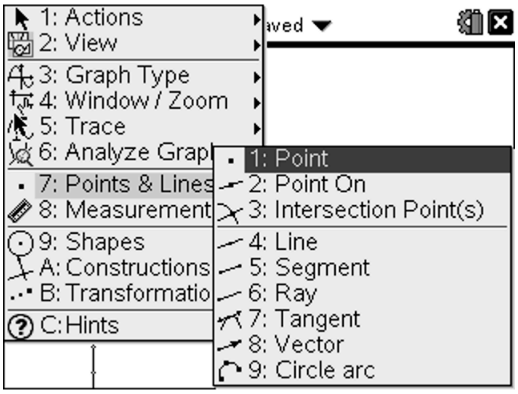


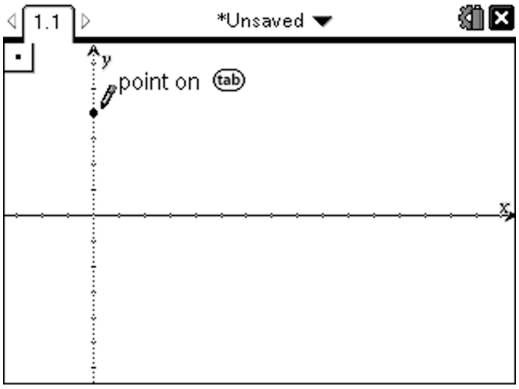
5. Move the cursor over **one of the notches** on the y-axis. Press (ctrl) then (menu) and select ‘Attributes’. Scroll down the four options to the last two, to Hide the Tic labels and Hide the End values.
Then grab the page and move it leftwards, to a position similar to that shown in the screenshot, below right.





6. Select the Point tool, and put the first point on the y-axis, **on** one of the notches, at (0,4)
[this will constrain that point to only move to integer values on the y-axis]



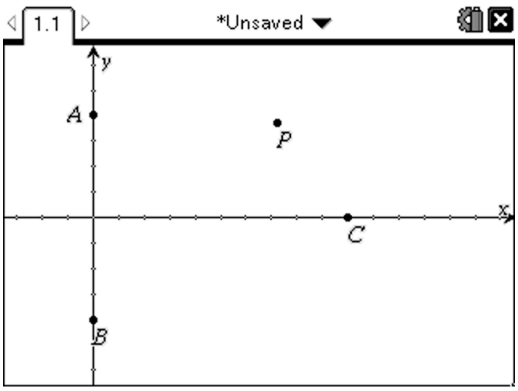


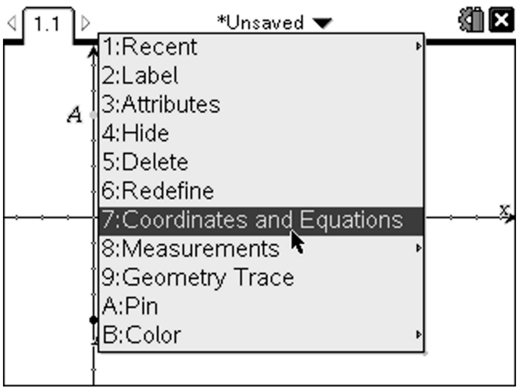
7. After you have placed the point, press (⇧shift) then [A] and the point will be labelled with a capital letter A.

8. Similarly, place points B and C **on** the axes notches at (0,-4) and (10,0), labelling each one as you go.

9. Place point P in empty space, away from the axes. *[see screen shot, below left]*

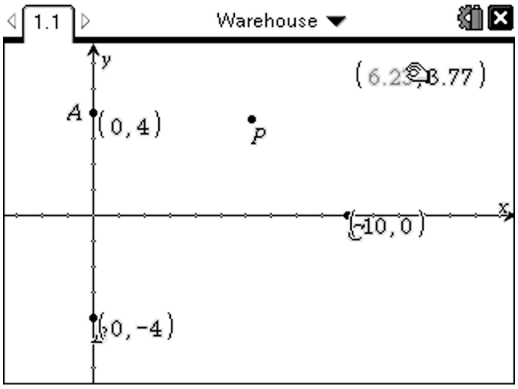
10. Move over each point, press (ctrl) then (menu) and select ‘Coordinates and Equations’ to have the coordinates of each point displayed.

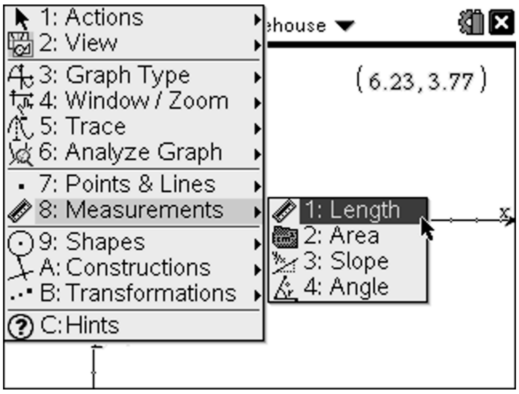




11. Grab the coordinates for point P, and move them to the top right of the screen.
[see screen shot, below left. This will stop the screen becoming too cluttered later on.]

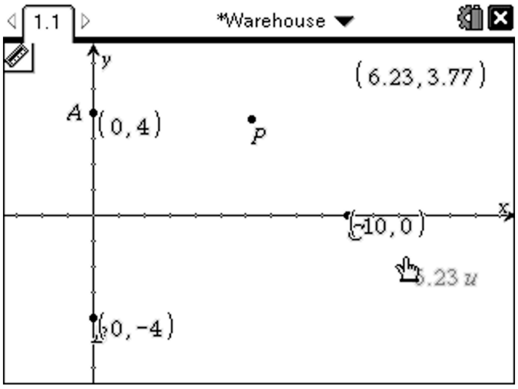
12. Select the (menu) > Measurements > Length tool.

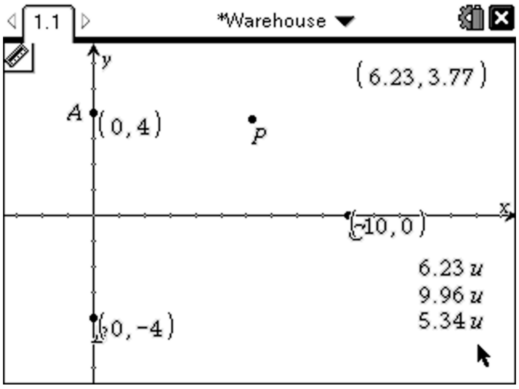




13. Click first on point P, then on point A, and then place the resulting length measurement bottom right.

14. Repeat the process, measuring P to B and P to C, placing their lengths below PA's, as shown below, and then press (esc) to exit out of measurement mode. *Your screen should look something like below, right.*

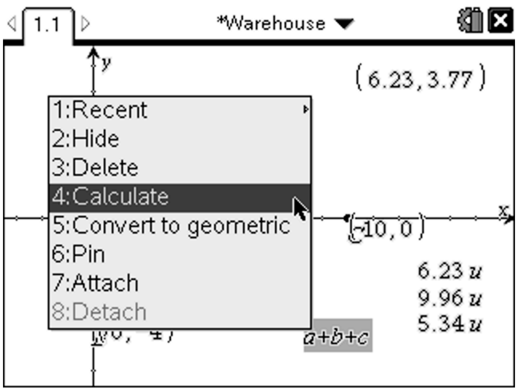


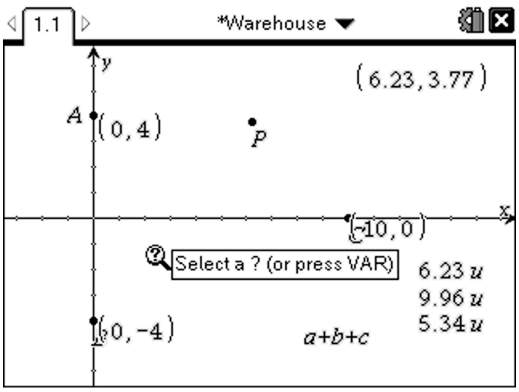


15. In order to add these lengths together, move into empty space and press (ctrl) then (menu) and select 'Text'.
Then type *a*+*b*+*c* into the text box that appears, and press (enter).

16. Move over the text, press (ctrl) then (menu) and select 'Calculate'.

17. Move the cursor about a bit using the touchpad, and you will be prompted for a value of *a*.
Move the cursor over the first length measurement that you took, and press (enter).

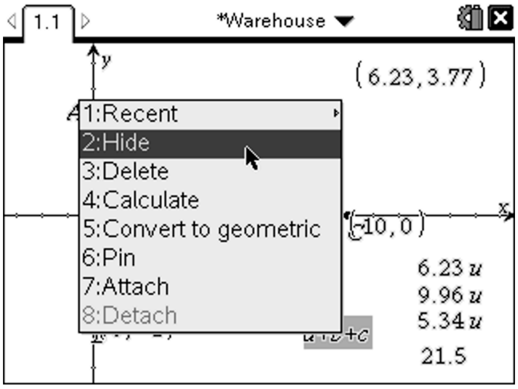


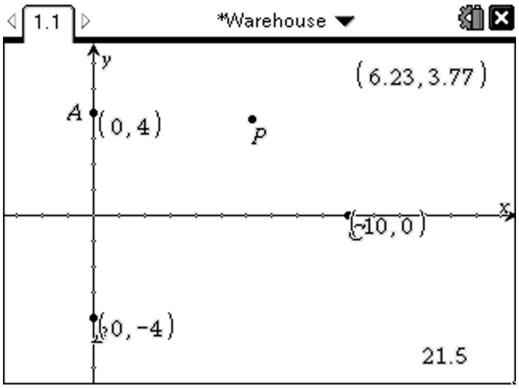


18. You will then be prompted for *b*.
Click on the second length measurement.
Then you will be prompted for *c*, so click on the third length measurement.

19. The answer to the calculation then appears.
    Press (enter) to place it on the screen.

20. Now we tidy up the screen, hiding the things we no longer need to see.
    Move over the text formula, press (ctrl) then (menu) and select 'Hide'.
    Repeat this hiding process for the three length measurements, leaving you with a screen like the one shown bottom right.
    And we're done!





21. Now spend some time grabbing and moving point P around, and see where it needs to be located to achieve the lowest value for the total length.

22. If you wish to increase the number of decimal places on display for the total length, move over the number, and press (+) several times.

Write down <u>any</u> observations that you have about where you think the best location for the warehouse P should be. At the moment, we are basing our conclusions around knowing that the shops are located at (0,4), (0,-4) and (10,0). We can change the locations of these points later, if we want.

Write down a conjecture about <u>why</u> you think the best location is where it is.

By now, you should have a good 'feel' for the problem and its likely solution.
We now need to look at it with more <u>rigour</u>........

Point P currently moves <u>anywhere</u> on the coordinate plane. This means it currently has '2 degrees of freedom'. In other words, we can independently choose an *x*-coordinate and a *y*-coordinate for P.
<u>This is too much freedom!</u>

Mathematicians like starting off solutions to problems with only <u>one</u> degree of freedom, or <u>one</u> variable only.

Now, read on.....

MODEL 1

To start off with, we shall assume that the best location for point P is somewhere on the actual x-axis.
So we now have P(x,0), A(0,4), B(0,-4) and C(10,0).
Does this assumption of putting P on the x-axis go against your conjecture that you wrote before?
[I hope not!]

So, if point P is at (x,0), let f(x) represent the value of the total distance PA+PB+PC.
f(x) is the total 'warehouse-to-shops' distance that depends upon the value of x.

Which of the following functions can be used for f(x)?
Write down any work that you do in order to make your selection:

i.) $f(x)=\sqrt{x^2+16}+\sqrt{x^2+16}+10-x$

ii.) $f(x)=2\sqrt{x^2+16}+10-x$

iii.) $f(x)=x+18$

iv.) $f(x)=2\sqrt{x^2+16}+\sqrt{(10-x)^2}$

v.) $f(x)=2\sqrt{x^2+16}+\sqrt{(x-10)^2}$

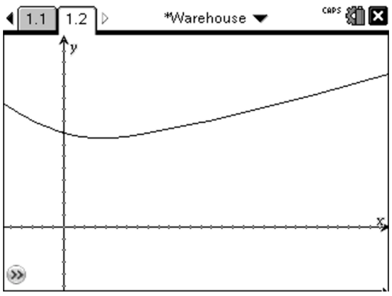
vi.) $f(x)=2\sqrt{x^2+16}+\sqrt{x^2+100}$

vii.) $f(x)=3x+18$

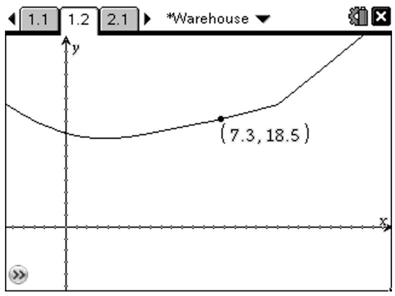
To graph one of the function(s) that you select, you need to insert a new Graph Page to your document.
Press (doc▾) to help you do this.

After typing in the name of the function, and pressing (enter), it will seem as if nothing has happened.
Think and then **write down** why this is.

With some adjustments to the screen settings, you should aim for something looking like one of the graphs below: *(note that some of the screen information has been deliberately hidden)*



or



Sketch a neat copy of whichever graph you obtained in your notes, and include the numerical values that you have at the extremes of your axes.

Now select the Point tool, and **place a point** anywhere on the function.
The coordinates of this point should be shown automatically.
Move the point along the function and see what happens at the minimum.
Write down what you see, what you find out and whether it agrees with your earlier work and conjecture(s).

You should now have a result for the Warehouse problem with A(0,4), B(0,-4) and C(10,0) and point P(x,0)
Does there appear to be **any** significant reason why the best location for the warehouse P is where it is?

If you can't conclude anything yet, it's not surprising, as you've only really looked at one situation!

Now spend time looking at 4 or 5 other situations, graphing the new functions that you will find yourself dealing with, and finding the location of P which minimises the function distance. You can revisit page 1.1 to experiment first with new coordinate points, by simply grabbing and moving them. Here are a couple of suggestions that you could try *(you can pick just one of them, or do a different one of your own)*

Suggestion 1:

A(0,4) B(0,-4) C(10,0)
A(0,5) B(0,-5) C(10,0)
A(0,3) B(0,-3) C(10,0)
A(0,2) B(0,-2) C(10,0)
etc.

Suggestion 2:

A(0,4) B(0,-4) C(10,0)
A(0,4) B(0,-4) C(9,0)
A(0,4) B(0,-4) C(8,0)
A(0,4) B(0,-4) C(7,0)
etc.

Note: we are keeping A & B to be equidistant from the x-axis, so that we can continue to assume that P lies on the x-axis, as before.

Keep a **written account** of all that you do (workings, results, algebra, graphs, etc.), and make sure that your results are clearly recorded.

Write down any concluding conjecture that you have, arising from your experiments.
Can you say anything yet about the best solution you'd expect to a warehouse situation of the form:
A(0,h), B(0,-h) and C(k,0)? (where h and k are integers)

Don't worry too much if you can't conclude anything that you are certain about yet.

Let's stop here for the moment, and try a new approach......

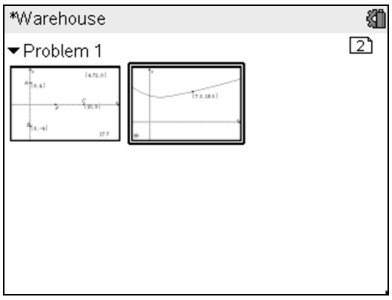
MODEL 2

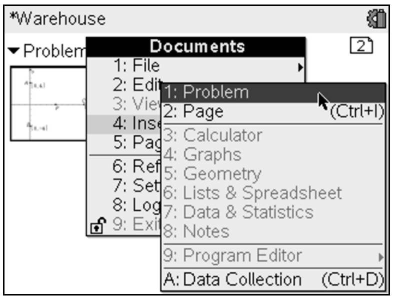
Our first model used coordinates to locate the position of the warehouse, P, on the *x*-axis.
Alternatively, we can describe its location in terms of an **angle**.
The angle we will choose is angle PAO (where O is the Origin)

To help you see this, we shall make a copy of page 1.1 and adapt it slightly.
Here's how we do that:

Press ctrl then ▲

Press doc▾ and select Insert > Problem

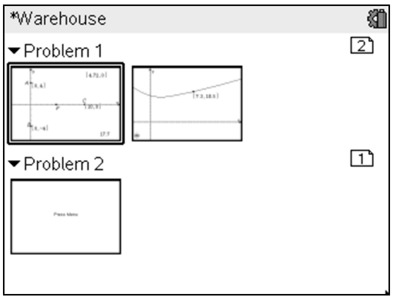


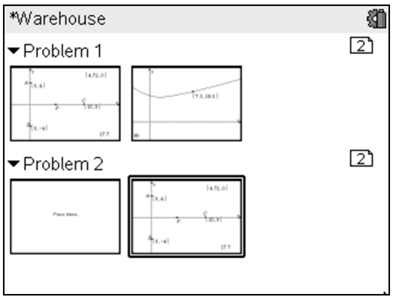


Press ▲ then ▲ to <u>highlight</u> page 1.1

Press ctrl then C to <u>copy</u> page 1.1

Move ▼ to Problem 2, and press ctrl then V to <u>paste</u>



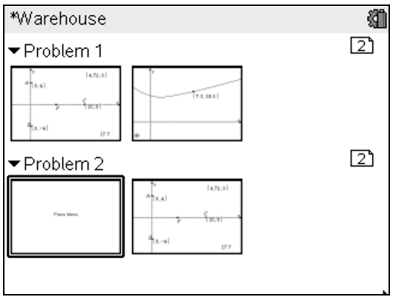


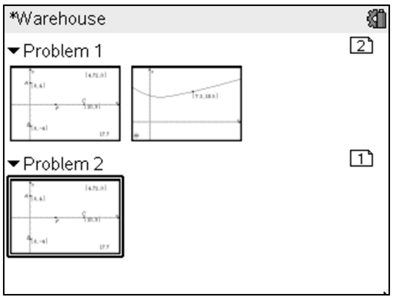
We now have an extra page in problem 2 that we no longer need, so....

Move ◀ to highlight page 2.1.

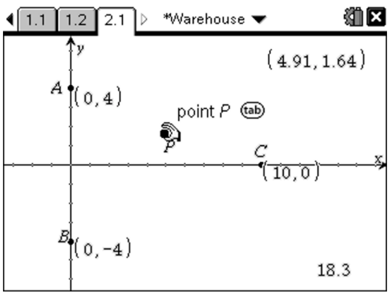
Press del to delete page 2.1

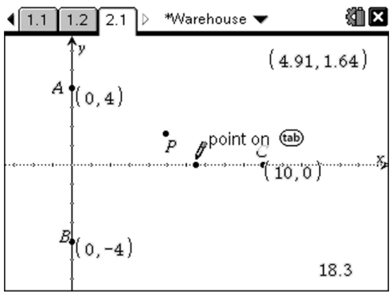
Press enter to bring the new page 2.1 to full screen.





Move over point P and press ctrl then menu and select 'Redefine'





You now have control over a new point that you should place on the *x*-axis, but <u>**not on**</u> one of the notches.
Place it <u>**between**</u> two notches.
This will allow it to freely move along the axis, and not be fixed to integer values.
Press enter to redefine point P to exist on the *x*-axis.
Move it back and forth to check that it behaves as you would expect.

Now for the angle.
Use (menu) > Points & Lines > Segment to draw a segment between points A and P
Use (menu) > Measurements > Angle and then select the three points of **P**, then **A** then the **O**rigin in that order.
The angle PAO should then be marked.
It is likely that the angle measurement says ‘rad’ after it.
This means that the angle is not measured in degrees, but in radians.
We want it measured in degrees.

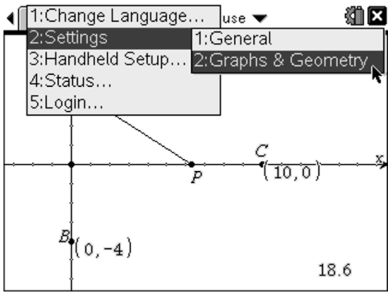
To change the angle measurement settings to degrees.....

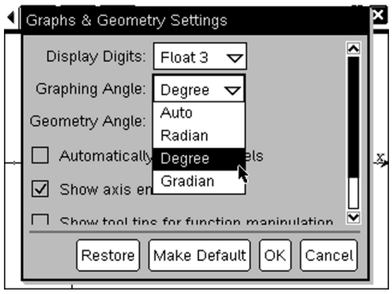
Click on the Cog/Battery icon in the top right corner.

Select Settings > Graphs & Geometry

Change Graphing Angle to ‘Degree’.

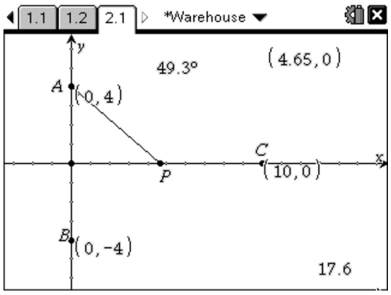
Click on OK.





The angle measurement is now in degrees.

You may like to grab and move the number for the angle measurement away from the coordinates, to keep things uncluttered:



If not done so already, move points A, B and C back to their original positions, shown above.
As before, move point P around and keep an eye on angle PAO.
Remember that you know how to increase the number of decimal places shown for any displayed number.
Write down anything you notice when the distance PA+PB+PC seems to be smallest.

Here comes the algebra and graphing bit again......

Point P is now defined by angle PAO’s size.
Let angle PAO = x
Let f(x) represent the value of the total distance PA+PB+PC.
f(x) is the total distance, that depends upon the value of angle x.

Draw a neat sketch of Nspire page 2.1, with all the required detail, and label angle PAO to be x.

i.) Use your diagram to convince yourself that $PA = \dfrac{4}{\cos x}$

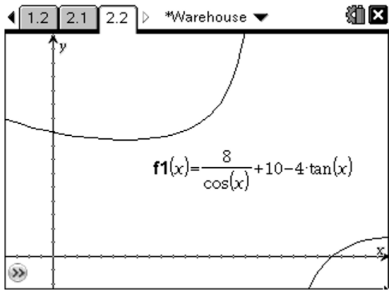
ii.) Now state the length of PB in terms of x.

iii.) What is the length of PO in terms of x?

iv.) Hence, go on to show that $f(x) = \dfrac{8}{\cos x} + 10 - 4\tan x$

Graphing the Function

Insert a new Graph Page, type in the function just derived and rescale it accordingly to give something like this:



As before, plot a point on the function and move it to the minimum.
Does this agree with your earlier experimentation?

It is now well worth sketching a plan view of the shops and warehouse for the situation where the angle PAO takes on this minimum value. **Fill in all the angles** that you can on your diagram, and it ought to reveal a nice symmetry for the optimal solution.

Do you think that this Model 2 is better than Model 1, or not? **Give a reason(s).**

The graph also seems to suggest two other things are happening.....

1. We have a section of the function to the left of the *y*-axis.
   This suggests that negative values of *x* can be used.
   Explain, with the use of diagram(s) if required, what a 'negative value of *x*' might look like in terms of the original diagram, with shops A, B and C and warehouse P.

2. There is also a second branch of the graph in the bottom-right-hand corner of the graphing screen.
   Part of the branch is above the *x*-axis and part is below.
   What values of *x* are generating this branch?
   Again **explain**, with the use of diagram(s) if required, what the situation looks like with these values of *x*.

Generalising Model 2
So far, you have used A(0,4), B(0,-4) and C(10,0).
Now alter the coordinates of A, B and/or C so that they remain of the form A(0,*h*), B(0,-*h*) and C(*k*,0)
Position point P at the optimum location on the *x*-axis to minimise the total distance PA+PB+PC
How does the result for the optimum angle compare to your previous answer?

Are you able yet to **generalise your conclusions** to where the warehouse should be located when dealing with a situation when A and B are equidistant from the origin?

<u>MODEL 3</u>
Model 3 is very similar to Model 2, in that an angle will be used as a variable to determine P's location.
In Model 2, we used angle PAO.
In Model 3, we will use angle APO.

Revisit the steps in Model 2 for how to Insert a New Problem, make a copy of page 2.1 and then delete the numerical value of angle PAO on your new page 3.1

Measure angle APO in a similar way to before, and then tidy up the screen if things are cluttered.

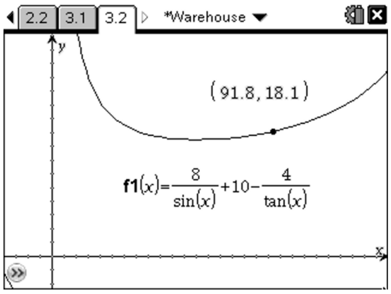
Move point P to its optimum position and note the value of angle APO.
Is this value consistent with your result from Model 2?

Define x to be the angle APO.

Draw a neat sketch of your Nspire page 3.1, with all the required detail, and label angle APO to be x.

In a similar manner to Model 2, **show that** for this situation $f(x) = \dfrac{8}{\sin x} + 10 - \dfrac{4}{\tan x}$

Either insert a new Graph Page and plot this function.
or you could copy page 2.2 to make page 3.2 and then edit the previous function from Model 2.



Moving a point on the graph to the minimum turning point of f(x), should support your earlier findings.
Explain, with the use of diagram(s) if required, why this gives the 'same' solution as before.

As before, sketch a plan view of the shops and warehouse for this optimum value of APO, filling in as many angles as you can. Again, does this reveal any symmetries present in the optimal solution?

In Model 2, we had negative values of x and separate branches of the graph to consider.
We don't appear to have the same issues with this Model.
Or do we?
Explore if there is anything else going on that we can't yet see.
Try rescaling the axes.
Explain, with the use of graph(s) and diagram(s) if required, anything that you do (or don't) discover!

MODEL 4
This is the model of your choice!
I suggest you read all of this section, before deciding how to progress.

So far, we have used Pythagoras' Theorem and Trigonometry (twice) to model the situation of 3 shops, where two of the shops are symmetrically positioned either side of the origin (shops A and B)

Where do you want to go next?
Which type of model do you think has most potential to help solve the next situation that you could create?

Instead of Pythagoras' Theorem or Trigonometry, you *could* tackle the Warehouse problem using '*just* geometry', by making use of items on the (menu) > Constructions menu.
This may be beyond what you know about, or want to experiment with.

Alternatively, you could look at a non-symmetrical situation.
For example, shops are located at A(0,4), B(0,0) and C(10,0)
Or maybe A(0,4), B(0,-4) and C(10,2)
It's unlikely that the warehouse will be optimally located on the *x*-axis for these.

What if there were fewer shops? What if there were more shops?

You can Insert a New Problem, and copy and paste any existing Nspire page into that new Problem, to experiment with one/some of these new situations.

Make sure that it's clear from your working **what it is** you are going to be looking at. Lots of random diagrams and algebra without an introductory sentence or two won't be easy to follow.

Whatever you decide to do, the advice is very much *'take it slowly, making only small adjustments to the situation that you already know about, and have already analysed'*

Acknowledgements
Thanks to Ron Lancaster, Senior Lecturer of Mathematics Education at the University of Toronto for the initial idea and introduction to the many approaches to this problem.