Examples of end Examination Tasks supported by Technology from Austrian Secondary Schools

Josef Böhm, T[3]-Austria, nojo.boehm@pgv.at

In addition to the examples presented in my talk A – B) I offer a choice of a couple of other examples /1 – 8) to be worked through and discussed in the workshop.
The examples are presented without proposed solutions. If you would like to receive the solutions then you are friendly invited to contact me (nojo.boehm@pgv.at).

Example A

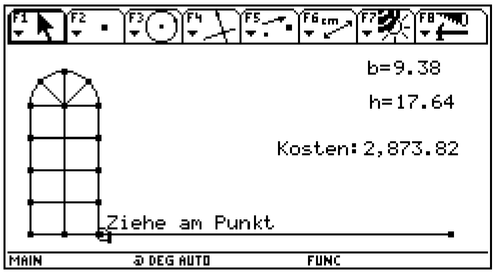
Given is a family of funktions $g_k(x) = \frac{1}{16k}x - \frac{1}{32k}x^2$.

a) Which is the form of all curves of the family?
b) What are the common properties of all graphs? Give reasons!
c) What is the influence of parameter k on form and position of the graphs. Present your findings using an appropriate survey.
d) Find a "partner family" $f_k(x)$ so that each g_k is intersecting the corresponding f_k orthogonally.
e) Show for any k that condition d) is fulfilled.
f) What is the common area enclosed by two "partners"? Shade this area on your device for any appropriate k. Explain how you can achieve this.
g) Which value of k makes this area extremal? Is it a Maximum or a Minimum?

Example B

A window has the form of a rectangle together with a semicircle (see the details in picture at the right).

(CU = Currency Units)



All straight connections cost 22 CU/dm, and the curved parts cost 25 CU/dm. Which measurements keep the production cost minimal?

a) Use the Dynamic Geometry model [1] to obtain an appropriate estimation applying an approximating function of your choice. Check first the correctness of the model.
Explain carfully your strategy (table, sketch, ...).
Additional Question: What is the reason that a polynomial function cannot fit exactly?
b) Solve the problem analytically using means of Calculus.
c) Show that the graph of the costfunction found in b) passes the data points produced in a) and answer the following question dependent on the choice of the independent variable (b or h):
c1) If b was chosen: What is the end behaviour of the curve? Find the approximating curve for the end behaviour and interpret this curve in connection with the problem.

c2) If h was chosen: Interpret the intercept of the cost function on the vertical coordinate axis..
d) How does the result change if one has to consider the cost for the glassing (8 CU/dm^2), too? Only the analytical treatment is necessary.

Example C

A company has to build an air filter system to avoid air pollution with total cost of 170 000 EURO. The company made savings for this investment during the last three years: at the end of each months it made a deposit of 1300 EURO, which gained 6.4% interests annually.

a) How much could be saved for this important investment during the three years?
How much outside capital – rounded to 100 EURO – is still necessary for this investment?

The company can choose between two financial models:

<u>1.Model</u> The company receives a subsidy from the government in an amount of 15% of the total cost. The remaining rest must be financed by a bank loan with a life of 15 years and an annual interest rate of 7.5%.

b) How much is the loan and how much are the monthly payments (due at the end of each period)?

c) What is the internal rate of return of this financial model considering a credit tax of 0.8%, a charge of 3% and the subsidy (accurate to 0.1%).
(Tax and charges are subtracted from the loan immediately).

<u>2.Model:</u> The company takes a loan which is subsidized by the government. The loan must be paid back within 10 years at a nominal annual interest rate is 6% with 4 compounding periods/year. The payments are due at the begin of each quarter and they consist of the constant amortization rate and the interests of the outstanding loan.

d) Give the first four and the last three rows of the amortization table, showing the number of the period, the outstanding loan, the amortization rate, the interests and the full quarterly payment.

e) Find a formula for the payment in any period k and compare with the respective values in the amortization table.

f) Subsidy of the government is paying 15% of the amortization rate. Complete the table by the values of the reduced payments. What is the formula for the reduced payments?

g) What is the effective interest rate of this loan considering the subsidy, a tax of 0.8% and credit charges of 1% (accurate to 0.1%)?

h) Use CAS to find a closed formula for the present value of these variable payments for a general "Abzinsungsfaktor" v? ($v = (1+i)^{-1}$ with i = interest rate).

Example D

1cm = 600m

In the region north of River Danube a golf course is planned. Its northern border is formed by a newly to build street which shall near *Winkl* (point W) continue the straight - also to be constructed - street S_1. Then it shall pass point O to finally near *Absdorf* (point A) run into the - also to be constructed - road S_2.

a) In first approximation find the line of construction of the northern border, which shall be described by a polynomial function of lowest degree. The connections between this road and S_1 and S_2 must be smooth.
The data must be read off from the map as accurate as possible. What is the function term?
Give reasons for your choice of the polynomial degree.

b) The connection between roads should not be only smooth - i.e. differentiable, but one has to avoid sudden changes of curvature, too. In order to reach this goal the curves must have the second derivatives in common.
Which polynomial function will describe the run of the road now. Use this function for the next tasks.

c) Draw by hands the new road on the map using at least 6 additional points.
Define the line S_1 - new road - S_2 as a piecewise defined function. Write down the correct syntax.

d) What is the difference in the lengths of both models?

e) Which area is available for the course if the southern border is formed by an existing road S (connecting road between W and A)? In 1st approximation you can use the airline between W and A.

f) What do you think? Is the real area greater or less that the value calculated? Try to improve your result in a not too difficult way. Describe shortly your choice of improvement.

Example 1

A cup of brass is produced by rotation of the hyperbola h: $x^2 - y^2 = 4$ and the parabolas:
$p_1 : y = \dfrac{5x^2}{16}$ *and* $p_2 : y = -a\,x^2 - 1$ for $-4 \le y \le 6$ around the y-axis.

a) Which is the complete equation of p_2, if the circular ring serving as base has a thickness of 1?

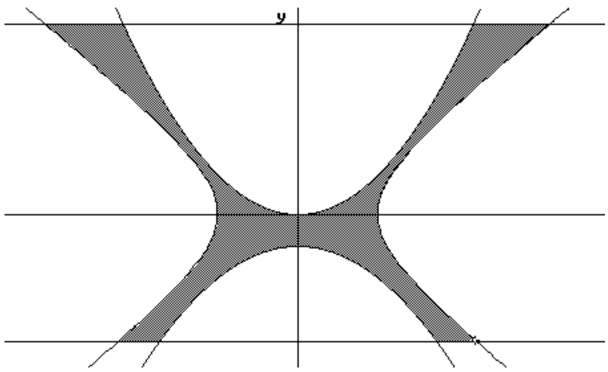
b) Use an appropriate tool and/or technique to present the intersection figur of the cup similar to the picture given below.Explain carefully what you are doing.

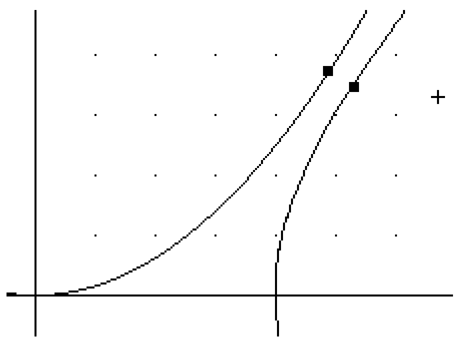
c) Which is the mass of the cup, if all measures are in cm and the specific weight of brass is 8.4?

d) How much liqid can be kept in the cup if it is filled and what is the height of the liqid, after having poured in 1/16 litre? Give reasons for the existence of two solutions and explain the second solution.

e) What is the area of the intersection?

f) What is the minimum thickness of the cup?





$$\mathrm{APPROX}\left(2\cdot\left(\int_{-4}^{6} k1\ dy - \int_{0}^{6} k2\ dy - \int_{-4}^{-1} k3\ dy\right)\right) = 19.94141583$$

Example 2

A cubic is given by its 2nd derivative $y'' = -\frac{2x}{3}$ and one point of its graph *P* (6/-6). The graph is symmetric wrt to the origin.

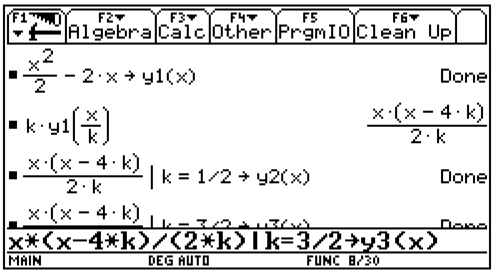
a) Find the cubic!

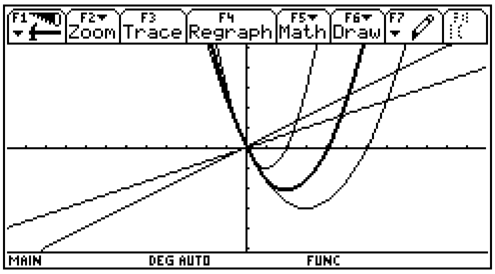
b) Sketch its graph for $-3\sqrt{2} \le x \le 3\sqrt{2}$. Reflect this graph wrt to $y = x$ and reflect the resulting figure wrt to the *y*-axis. Give a precise report how to create the graphs on the screen.

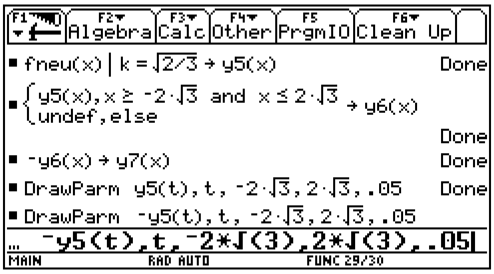
c) What does the figure look like?

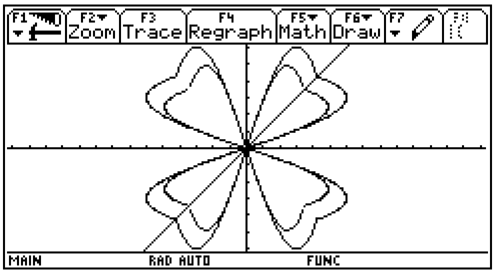
d) This figure is circumscribed by a square. How many percents of the area of the square is filled by the figure?

e) The function found in a) shall be changed by a compression or extension with the origin as its center so that the resulting figure which is similar to c) is covering one third of the area of the square. Which is the new function term?









Example 3

A mathematics teacher needs a new example for his end examination. He wants to use a function which has the form

$$y(x) = \frac{a\,x^2 + b}{c\,x^2 + d},$$

and shows the following features:

- its graph has poles at $\pm\sqrt{2}$,
- a line parallel to the x-axis in a distance of $\frac{3}{2}$ is an asymptote of the graph,
- its y-intercept is 3 and
- one of its zeros is at $x = 2$.

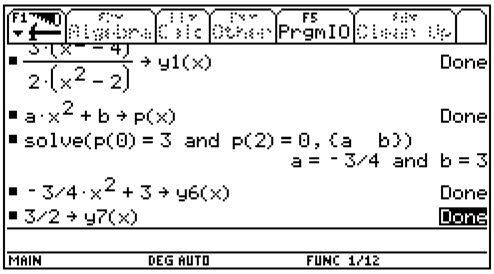
a) What is an appropriate function term? Why does the system of equations deliver infinitely many solutions?

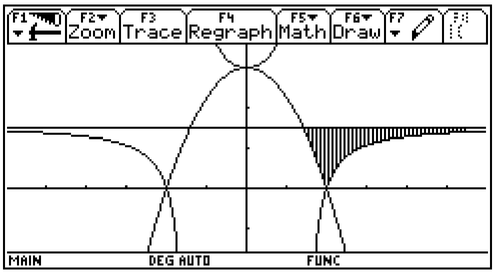
b) Sketch the graph and label its important properties
If working with the *TI*-92: How can you obtain a graph on the screen without the nasty vertical lines?

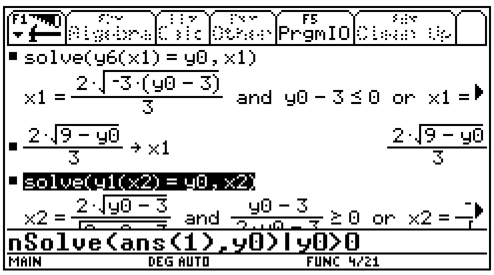
c) Find a parabola which has the extremal value and the zeros in common with the given graph..

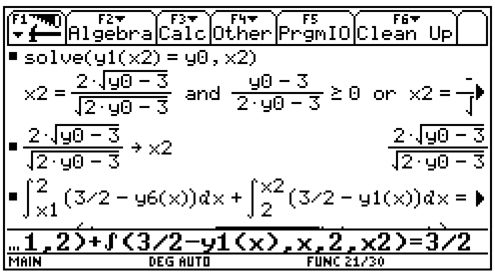
d) The horizontal asymptote and both graphs form right of the right pole a common area which reaches to infinity. Investigate whether this area converges to a finite value or not.

e) How far must the asymptote be translated downwards that this area gets the value $A = 1.5$?
Solve the resulting equation at least in two ways accurate to 2 decimal digits. (Apply atmost in one case the solve-command.)









Example 4

There are many good reasons to model a certain cost function by a quadratic function. One knows the proportinal cost with 1950*CU* (curreny units). But completely known is the demand-function

$$p_N(x) = \frac{795000}{x+170} - 300.$$

a) What are saturation amount and maximum price?

b) Using this model estimate the maximum revenue.

c) The supply function is given by $p_A(x) = 22{,}15x + 750$. The estimate for the demand at the beginning is 100*SU* (Selling Units) and fixes the selling price according the demand function. How will price and amount develop in the next future? Describe the development for the next three market periods. Use an appropriate tool to give a long-term forecast. Will we have a market equilibrium at any time?

d) What is the complete modeling cost function if one knows the total cost for 40*SU* with 120 000*CU* and if one assumes that the maximum profit can be made by selling 30*SU*?

e) Compare the maximum profit with the profit gained by selling the amount with minimum average cost.

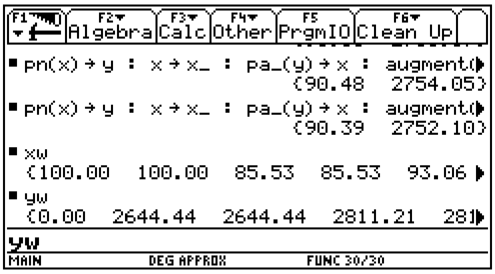
f) Compare the elasticity for the revenue maximum amount and the profit maximum amount. Interpret the results.

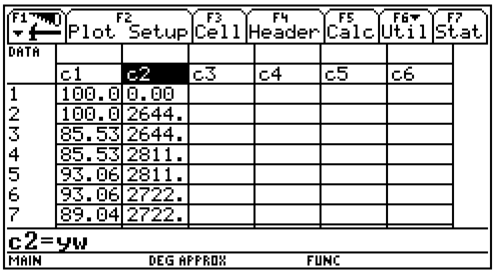
g) Where is the Break-Even-Point?

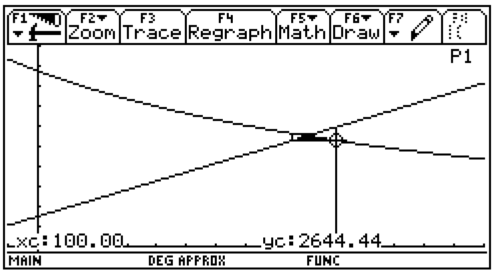
h) Find the consumer and producer surpluses.

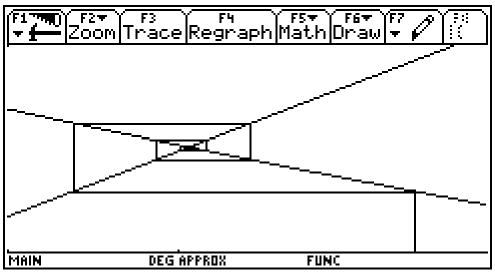
i) Present all your findings in one or more graphs.

```
100→x:0→y:{x}→xw:{y}→yw
pn(x)→y:x→x_:pa_(y)→x:augment(xw,{x_,x})→xw:augment(yw,{y,y})→yw:{x,y}
```









Example 5
(Josef Lechner, BG Amstetten)

a) The acceleration acting on a parachutist is equal to the acceleration of earth minus the reverse acting brake acceleration caused by the air which is proportional to the square of the velocity (prop. factor $b = 0.006$).
Set up the respective differential equation and give the solution as $v(t)$ with $v(0) = 0$.
($g \approx 10\text{m/s}^2$; work in Exact Mode).

b) Find the respective *time-distance*-function with $s(0) = 0$.
How far has the parachutist fallen until the parachute opens if he opens the parachute after 5 seconds. Which is his falling velocity in this moment?
Show that $s(t) = \frac{\ln(\cosh\sqrt{b \cdot g \cdot t})}{b}$ also describes the distance falling down.

c) Which end velocity can be reached? Using the *t*-*s*-function find a relation for the end velocity.
Represent the acceleration as a function of time and find its limit.
Give a physical interpretation of the run of acceleration.

d) Describe the run of velocity using a limited growth function. Choose an appropriate prop. factor that the function found matches the velocity function from above as accurate as possible.

e) Extra credit: Increasing altitude causes a decrease in air density. which influences even parachuting. Starting in an altitude of 2000m the influence of air density can be modelled by
$b(s) = 0.003 \cdot 0.95 \frac{2000 - x}{100}$.
Solve the differential equation approximatively in DE-Mode and sketch the resulting velocity function.

Example 6
(Eleonore Eisler, BHAK Tulln)

Given are the Tuesday- and Thursday prices of the ATX, DAX and Dow Jones on the Vienna Stock Exchange (Wiener Börse).

Datum	ATX	DAX	Dow Jones
2.1.2001	1070,44	6394,77	10750,26
4.1.2001	1068,43	6403,88	10954,80
9.1.2001	1077,64	6384,94	10640,05
11.1.2001	1081,22	6406,38	10578,55
16.1.2001	1072,95	6308,27	10572,55
18.1.2001	1091,03	6603,62	10601,83
23.1.2001	1090,13	6659,96	10578,24
25.1.2001	1087,13	6767,01	10646,97
30.1.2001	1091,79	6742,75	10719,09

a) Calculate the volatility of ATX, DAX and Dow Jones and give comments on the results.
(Volatility is the standard deviation of the internal rates of return from one price to the next one.)

b) Is there a correlation between
   b1) ATX-DAX
   b2) ATX – Dow
   b3) DAX – Dow?

c) Sketch the charts of all stock indeces and give comments.

Example 7

Given is a family of curves $y_a(x) = -\frac{x^4}{4a} + \frac{x^2}{2} + \frac{3a}{4}$; $a > 0$.

a) Investigate the family in general on symmetry, zeros, maximum- and minimum values and inflection points. Find the locus of extremal values and inflection points.
What is the difference between this family and the family with $a < 0$?

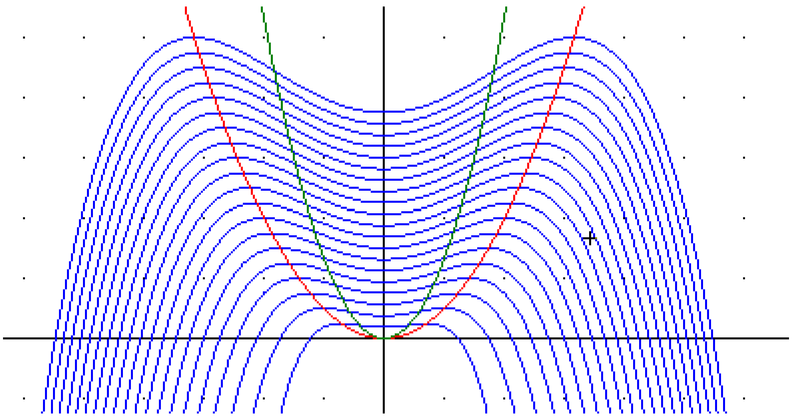
b) Which value for a yields an area between the graph and the x-axis $A = 100$?

c) The line passing both inflection points of the curves forms three distinct areas with the graph. What is the ratio of the measures of the areas? (For all curves of the whole family)

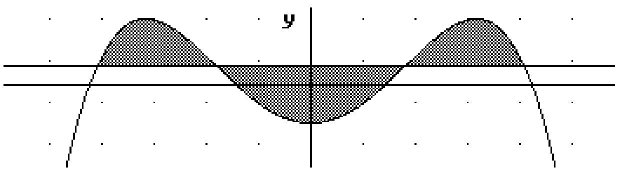
d) Which is the distance from the x-axis to lay a horizontal line that the three segments between the intersection points with the graph are of equal length?

e) Verify the results of (c) and (d) for $a = 10$.

f) Suppose there is another general function $y_b(x)$ with $b > 0$ and $a \neq b$. Give reasons why both graphs can never have real intersection points.



$$\left(f(x, 10) \ge y \ge \frac{80}{9} \wedge -\frac{\sqrt{150}}{3} \le x \le -\frac{\sqrt{30}}{3}\right) \vee \left(f(x, 10) \ge y \ge \frac{80}{9} \wedge \frac{\sqrt{30}}{3} \le x \le \frac{\sqrt{150}}{3}\right) \vee \left(f(x, 10) \le y \le \frac{80}{9} \wedge -\frac{\sqrt{30}}{3} \le x \le \frac{\sqrt{30}}{3}\right)$$



Example 8
(MarieLuise Müller, BHAK Waidhofen/Thaya)

A new product is to be introduced on the market. In the first week 120 units are sold (nmin = 1). It is supposed to have increasing selling numbers, but the company is also aware of the fact, that there is only a limited capacity for the goods on the market. 230 units can be sold in week #11 and 10 weeks later the selling number increases by 70 units.

a) Why can we in this case use the model of "limited growth"?. Describe the weekly growth using a recursive model and choose among {3%, 5%, 6%} for parameter p and {200, 300, 400} for parameter K (= capacity) to find the best fitting model.

b) Which number of goods can be sold after one year according to this model? When will we reach the half capacity?

c) The limited growth can be described by the formula $B(t) = K - (K - B_0)\, e^{-ct}$. Try to find an appropriate formula for our problem and find answers for task (b).

d) Give the sequence for the weekly increases and find out the time when this will fall below 5 units.

e) Give a sketch of the growth function together with the fivefold increases for the first two years. Take care for an accurate labelling of the axes.

Introduction of another product starts with a promotion campaign giving away 20 units (nmin = 0). Here again a limited market capacity is assumed (1500 units). The company believes that this product will be spread over the market like an infectious sickness, i.e. each possible customer purchases only once this product. A weekly growth rate of $p =$ 0.01% and logistic growth behaviour is expected.

f) What is the recursion formula for the logistic model. Give also the sequence of the fivefold weekly increases. Give a sketch of both sequences + labelling of the axes.

g) When will be reached half of the full market capacity? When will be reached the first 1000 of sold units? When will be reached the maximum increase and how much is it?

h) What are the answers to (g) if the company would start with giving away 50 units?

i) The closed formula for continuous logistic growth is $B(t) = \dfrac{K}{1 + \left(\dfrac{k}{B_0} - 1 \right) e^{-cKt}}$.

Use the sequence model to find an approximative value for c.

j) Use the analytic formula from (i) to check your answers given in (g).