A new CAS-touch with touching problems

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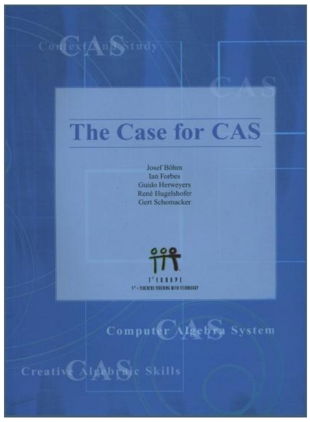
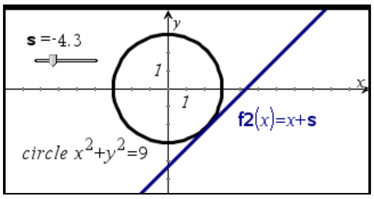


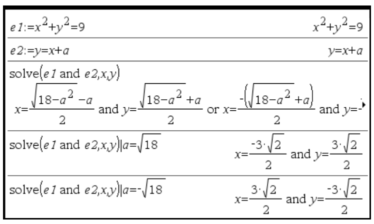
Parameters provide Maths with a new dynamic and lead sometimes to astonishing solutions that are not possible by hand.
In our book “The Case for CAS” we made a first approach to a dynamic algebra based on parameters. You can get it for free on www.t3ww.org. This book is a collection of nice CAS examples (for TI-89, V200), which can easily adapted to TI-nspire CAS, as the Math for these examples hasn’t changed. I like examples that are universal and not prepared for certain features of particular software.

For those who didn’t attend my first workshop, I would like to mention our newest book about “Quadratic Functions and Equations”. You have got the book translated to Flemish by Guido Herweyers. You can also have the German book for free (www.ti-unterrichtsmaterialien.net). On the same website you can also have all the tns-files for students and solutions for teachers, which show them the use of CAS and the advantages in Math teaching.
This book emphasizes an important concept of a dynamic Math based on parameters. Quadratics hereby play a crucial role, as quadratic equations are the only ones solvable with arbitrary parameters using an elementary formula.

In the first workshop I presented an important method, which I call „tangent method“. This method will play a central role in this workshop too.
With the tangent method I will show you how you can even find characteristics of conics in an elementary way.

Tangent method with a circle

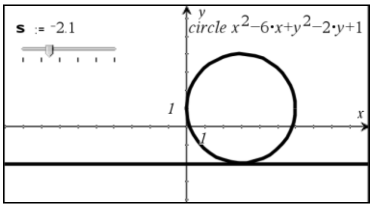


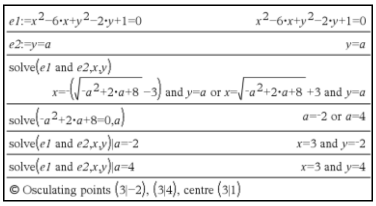


We first draw a circle $x^2+y^2=9$. You can draw the unit circle with $f1(x) = zeros(x^2 + y^2 - 9, y)$.
Draw a sheaf of lines $y = x + s$ and move the line with a slider for s.
There are three possibilities: two intersection points , one or none.
The dual solution corresponds with the tangent to the circle.
The geometric approach gives you no insight to the problem.
Therefore intersect the circle algebraically with the straight line $y = x + a$ (you have to take a new parameter or a new problem). The dual solutions lead to the tangents of the circle (see screenshot).

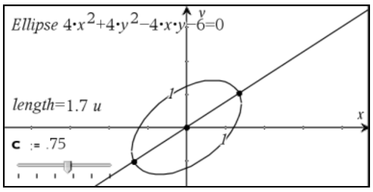
Let’s use the tangent method to determine the centre point of a generally positioned circle e.g. $x^2 - 6x + y^2 - 2y + 1 = 0$.

We can take the simplest tangents of the form $y = a$ (or $x = a$) and we easily get the tangents with the dual solution and the centre point $C = (3|1)$ as well as the radius 3 of the circle. (Compare with the method of completing the square.)

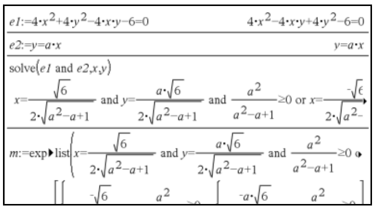


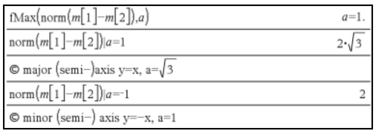


Now let us investigate generally positioned ellipses: I’m going to show you that even with generally positioned ellipses you can apply the tangent method. Even more: you can also determine the axis of such an ellipse.

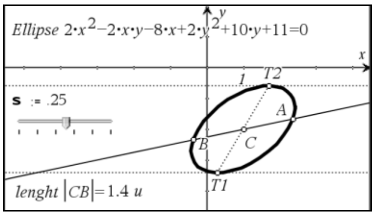


In a first step we take an ellipse $4x^2 + 4y^2 - 4xy - 6 = 0$ rotated with centre $(0|0)$. Intersect this ellipse with the line $y = ax$ and take the maximum distance from the intersection points (with fmax) and you get the parameter *a* for the major axis and also for the minor axis (perpendicular). Result: a=1. Length of the axis $2\sqrt{3}$ and 2.





And if the centre is not $(0|0)$? Let’s take as an example $2x^2 - 2xy - 8x + 2y^2 + 10y + 11 = 0$. Again the ellipse can be drawn with $\text{zeros}(2x^2 - 2xy - 8x + 2y^2 + 10y + 11, y)$.
How can we find the centre of the ellipse? With the tangent method calculate the tangent points T1 and T2 of the two horizontal tangents (any pair of parallel tangents would be fine as well). The segment from T1 to T2 is a diameter of the ellipse. The midpoint of this segment is the centre C of the ellipse. Result: $C = (1|-2)$.
The major and minor axis can be found as shown above.



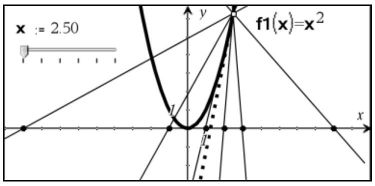
Conics

We can treat conics in a general position within the subject of quadratic functions and equations in an elementary way.
The general algebraic equation of order 2 is of the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$
Except for degenerated cases there are 3 possibilities

- a parabola if $ac - b^2 = 0$
- an ellipse if $ac - b^2 > 0$
- a hyperbola if $ac - b^2 < 0$

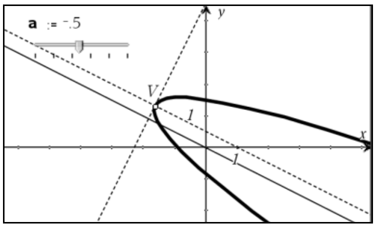
The students don’t have to know this theory at this level. Either you just let them try drawing such implicit functions. Or if you want a restriction to parabolas, just tell them to find and draw an equation of the above form with $ac - b^2 = 0$. Every student will have then his own individual ellipse.

Glance at Infinity

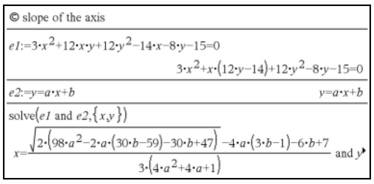


Do you want to join me on a Mathematical trip through the space? We draw a parabola and a sheaf of straight lines through a (movable) point on the parabola and the tangent in this point. Before x=100 the straight lines appear to be parallel. But they still meet at the common point. I.e. parallel lines meet in a point at “infinity”, like rails that seem to meet on the horizon (compare with the theory of projective geometry).
Of course the tangent dislodges more and more from the origin, but becomes more and more vertical. The vertical direction therefore defines the infinitely remote point, where all the parallel lines come together.

General Parabola



Let now students choose their own parabola with condition $ac - b^2 = 0$.
Example: $3x^2 + 12xy + 12y^2 - 14x - 8y - 15 = 0$ where $3 \cdot 12 = 36$.
Intersect the parabola with $y = a \cdot x$. Explore the impact of the parameter a on the intersection points graphically. The special case is $a = -\frac{1}{2}$ (parallel to the axis), where you have an intersection point at infinity.


© slope of the axis

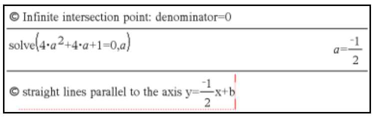
$e1:=3\cdot x^2+12\cdot x\cdot y+12\cdot y^2-14\cdot x-8\cdot y-15=0$ → $3\cdot x^2+x\cdot(12\cdot y-14)+12\cdot y^2-8\cdot y-15=0$

$e2:=y=a\cdot x+b$ → $y=a\cdot x+b$

solve(e1 and e2,{x,y})

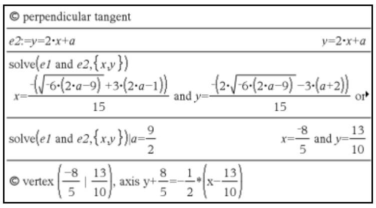
$x=\frac{\sqrt{2\cdot(98\cdot a^2-2\cdot a\cdot(30\cdot b-59)-30\cdot b+47)}-4\cdot a\cdot(3\cdot b-1)-6\cdot b+7}{3\cdot(4\cdot a^2+4\cdot a+1)}$ and y▸

Next intersect $y = a \cdot x + b$ with the parabola algebraically. The intersection point at infinity is the same for all these parallel lines (slope with denominator = 0). This “point” can therefore be interpreted as the second intersection point if $a = -\frac{1}{2}$ and all parallel lines with this slope intersect at this infinite “point”. As the axis of the parabola has the slope $-\frac{1}{2}$ the tangent in the vertex point has the slope 2. Now intersect the sheaf of lines $y = 2x + a$ with the parabola and find the dual solution, which gives you the vertex point $\left(-\frac{8}{5} \mid \frac{13}{10}\right)$ and at the same time the axis of the parabola $y - \frac{13}{10} = \frac{1}{2}(x + \frac{8}{5})$.


© Infinite intersection point: denominator=0

solve$(4\cdot a^2+4\cdot a+1=0,a)$ → $a=\frac{-1}{2}$

© straight lines parallel to the axis $y=\frac{-1}{2}x+b$


© perpendicular tangent

$e2:=y=2\cdot x+a$ → $y=2\cdot x+a$

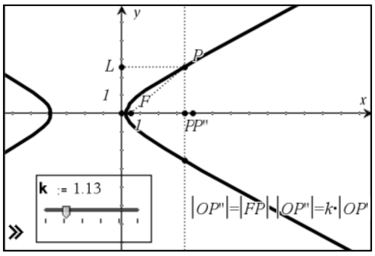
solve(e1 and e2,{x,y})

$x=\frac{-(\sqrt{-6\cdot(2\cdot a-9)}+3\cdot(2\cdot a-1))}{15}$ and $y=\frac{-(2\cdot\sqrt{-6\cdot(2\cdot a-9)}-3\cdot(a+2))}{15}$ or▸

solve(e1 and e2,{x,y})|$a=\frac{9}{2}$ → $x=\frac{-8}{5}$ and $y=\frac{13}{10}$

© vertex $\left(\frac{-8}{5} \mid \frac{13}{10}\right)$, axis $y+\frac{8}{5}=-\frac{1}{2}*\left(x-\frac{13}{10}\right)$

Transition between conics



As you have seen in my first workshop: ellipses are just compressed circles. Let’s see what happens if we take an ellipse on a trip to infinity?
The image shows the construction of conics from the guideline and the focus point (exercise in the 1st chapter of our booklet). With a slider you can show the transition from an ellipse to a parabola (osculation at infinity) to a hyperbola (an ellipse which intersects in an infinite remote point in the direction of the asymptotes).

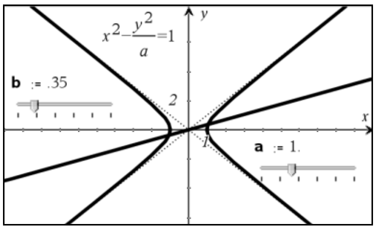
Hyperbolas

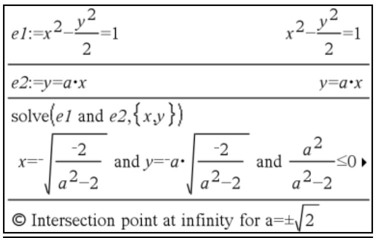
Parabolas are a special case. But hyperbolas can be considered like stretched ellipses (beyond infinity), therefore you can treat them in the same way as ellipses.

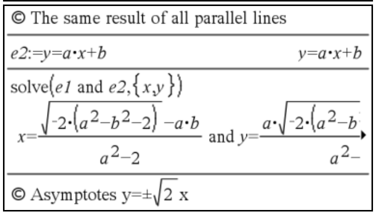
Let's first have a look at a hyperbola with centre point $(0|0)$ and axis parallel to the x-axis. Take as an example $x^2 - \frac{y^2}{2} = 1$.

First explore graphically the intersection points of $y = a \cdot x$ with the hyperbola. There are either two intersection points or none. The geometric investigation gives no insight into what happens at the transition between the two cases. Investigate this algebraically and intersect the more general case of sheaf of lines $y = a \cdot x + b$ with the parabola.

The zeros $a = \pm\sqrt{2}$ of the denominator determine the limit case where the intersection points tend to an "infinite point". The lines with the slopes $a = \pm\sqrt{2}$ can be interpreted as the asymptotes of the hyperbola (intersection point at infinity).







General position of the hyperbola

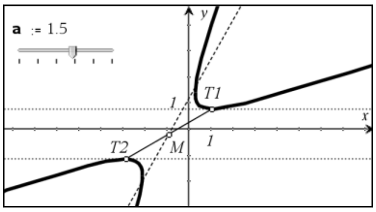
Have the students find their own hyperbola by choosing a quadratic form with $a \cdot c - b^2 < 0$. Example:

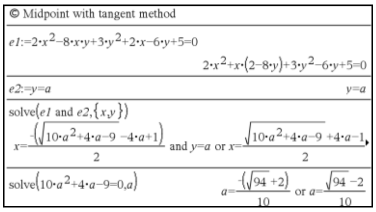
$2x^2 - 8xy + 3y^2 + 2x - 6y + 5 = 0$.

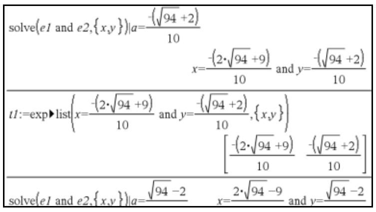
First determine the centre point as shown with the ellipse: Intersect the hyperbola with the horizontal line $y = c$ (or $x = c$) and calculate the midpoint M of the osculating points T1 and T2 of the two tangents (as shown above with ellipses).

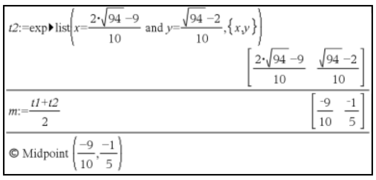
The result is $M = \left(-\frac{9}{10} \,|\, -\frac{1}{5}\right)$.

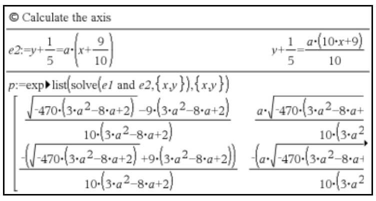
The line $y + \frac{1}{5} = a \cdot (x + \frac{9}{10})$ through M intersects the hyperbola in the points P1 and P2. Take the minimum of |MP2| and you obtain the slope of the axis of the hyperbola. Moreover, if you take the zeros of the denominator in P1 and P2 you will get the slope of the asymptotes (algebraic solution see tns-files)

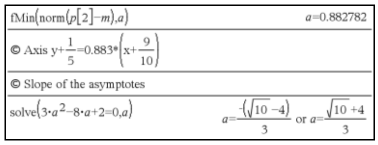




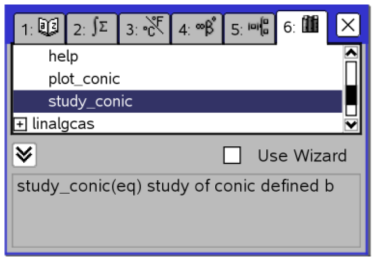








I hope you like this elementary way to the characteristics of conics.



In the conic-library (MyLib) you can find a procedure called "study_conics" where you can get the characteristics of conics as a black-box (wich needs much theoretical know how in Analytical Geometry about Eigenvalues and Eigenvectors).

With CAS you can do activities in a elementary way that are not imaginable and not possible with calculations by hand. Just try things and you will get astonishing results.

If you want to have this presentation and the tns-files, send me an e-mail.