An introductory calculus course with differential equations and dynamical systems at its center, a dynamic approach with TI-Nspire-CAS

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Abstract

A large majority of applications of calculus in different sciences consists of models with systems of differential equations e.g. dynamical systems, as any perusal of the current literature will indicate. In addition, as Henri Poincaré showed, the fundamental laws of nature are, from a mathematical point of view, differential equations [1]. Therefore, in our opinion, differential equations and dynamical systems belong at the center of calculus. Thanks to new technologies like TI-Nspire-CAS it is possible to solve differential equations numerically and graphically, so they can take their rightful place in an introductory course. The development of conceptual understanding, not algebraic technique is important. We designed an introductory calculus course based on these guidelines [2]. We will show, how this course is structured. It begins with a simple model (Newton's law of cooling) which leads -at first more intuitively- to the fundamental concepts of derivative (rate of change), differential equation, Euler's Method and model. The central chapters are dedicated to differential equations and dynamical systems, introducing the graphical method of state plane analysis. With the TI-Nspire technology it is possible to represent direction fields and trajectories and thanks to the sliders it is even possible to experimentally investigate the influence of the variation of the parameters of the systems. This can enhance a deep understanding of the systems under consideration. The last chapter discusses some basic algebraic methods of solving differential equations. The equation of the first chapter can eventually be solved exactly in a closed form. The usual topics of a classical calculus course are also discussed, but less extensively. With the focus on dynamical systems the great significance of calculus can be far better demonstrated than with the classical approach.

1 Introduction: Why do Differential Equations and Dynamical Systems belong at the center of a calculus course?

Why do we teach calculus in schools of higher general education? Calculus is one of the most important mathematical theories ever developed and it plays an important role in many scientific and technological contexts. Physics and engineering without calculus are simply impossible, nearly every scientist and every engineer uses methods and results from calculus in his work. From a utilitarian perspective it is very important that students of a lot of diverse disciplines have a basic knowledge of and basic competences in calculus. Moreover, calculus is one of the greatest achievements of the human mind and it is a language and a tool of modern science. It is impossible to understand the laws of physics without knowledge and comprehension of calculus and if one does not understand physics, one cannot understand nature. As Galileo Galilei told us: *Philosophy is written in that great book which ever lies before our eyes - I mean the universe - but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.* At the time of Galilei calculus was not yet invented. If calculus had been known by Galilei, he surely would have said *"derivatives, integrals, differential equations, phase planes and trajectories etc. "* instead of *"triangles, circles and other geometrical figures"*. But let us have a closer look at the way calculus is used by considering some important examples:

- **Public Health**: Every couple of years we have an epidemic like avian flu, swine flu etc. and an ordinary influenza occurs every winter. For a public health office it is very important to understand the way a disease spreads through the population and to predict what fraction falls ill and when. Furthermore, it is important to investigate the effect of different possible measures (vaccination, quarantine, drugs etc.). To this purpose, a model will be created. The model which is generally used is the S-I-R-model or a modification thereof. This model consists of a system of coupled nonlinear differential equations as we will see in more detail later in section 4.

- **Climate, Weather report**: Our daily weather report as well as the predictions concerning the climate are based on huge systems of differential equations. The most powerful computers of the world are permanently solving this equations with changing initial conditions.

- **Biology, population dynamics**: Predator-prey, symbiosis, and competition are just three different fundamental modes of interaction between populations in biology. Differential equations allow the modelling of these basic forms of interaction as well as more complicated systems with more than two species involved.
  As an example a political implication: Should we completely forbid the hunting of whales? Only a careful investigation of an appropriate model could provide sufficient data for evaluating the impact of whale hunting on the ecological system of the oceans. Thus calculus, and especially differential equations, are a necessary tool for assuming responsibility and making informed decisions.

- **Economics** All kinds of dynamical systems are used to investigate and predict economical systems.

- **Physics** Physics and calculus are strongly related. Physics prompted the development of calculus and calculus enabled the development of physics. Henri Poincaré formulated this very important connection in his famous paper "the value of science" as follows: **Newton has shown us that a law is only a necessary relation between the present state of the world and its immediately subsequent state. All the other laws since discovered are nothing else; they are in sum, differential equations** [1].

These applications differ fundamentally from the applications of calculus in an ordinary introductory course. The traditional highlight of application usually consists of the determination of optimal shapes, for instance the optimal shape of a box or of a beer can.
Usually, differential equations have no place or only a very small place at the introductory level. Traditionally, differential equations is a more advanced topic, requiring one or two years of calculus as a prerequisite. It is divided into a number of cases and subcases, and an array of different techniques is developed to deal with different cases. Some of these techniques are very elegant and clever, others are very sophisticated. With these techniques mathematicians learn the intricate reasoning and analysis which is fundamental for mathematics at a higher level, but this is in most cases too sophisticated and demanding for beginners of calculus. Technology is now providing the possibility to treat differential equations and systems of differential equations with a simple and intuitively clear numerical approach. The underlying concepts are comprehensible at this level. So it is possible, at an elementary level, to address problems and concepts which the overwhelming majority of learners would never get to see in a traditional curriculum. Problems with a much larger scope and importance than the traditional problems.

2 The design of the course, guidelines

- Calculus is a language as well as a tool for exploring dynamical processes in science. Students should be able to read and write this language. The techniques of calculus, the manipulative skills, must be subordinate to an overall view of the central concepts.
- The study of modeling inevitably leads to differential equations. Differential equations should therefore be fundamental objects of study.
- The concept of a dynamical system is central to science. Technology makes it possible to explore dynamical systems at the introductory level, using a simple intuitively clear numerical approach: Euler's Method.
- Processes of successive approximation are key tools of calculus. These processes are more important than the output -the limit- which often cannot be given in a closed form.
- Technological tools tremendously enlarge the range of questions and problems we can explore and the ways we can treat them. The appropriate use of these tools is therefore a central topic in a contemporary calculus course.

- Multiple representations: whenever possible, each topic will be representated and discussed from a numerical, a graphical and an algebraical point of view. The transition between these different modes of representation is of great importance and a software which enables these transitions is of great value.

- Euler's method plays the role of an universal mathematical and didactical tool in this course [3]. It makes it possible to solve differential equations numerically, but moreover it also provides the insight that an initial value problem has a unique solution, enables the definition of the number e and last but not least, can be used for an elegant proof of the fundamental theorem of calculus.

This course will result in substantial shifts in emphasis in comparison with a traditional course. Here are the most striking:

Increase	Decrease
concepts	manipulative skills
numerical solutions	algebraical closed-form solutions
graphs and graphical solutions	formulas
approximations	exact solutions

3 A context-orientated introduction to calculus

Nearly every introduction to calculus starts with the investigation of a tangent line, commonly a tangent line at a point of a quadratic parabola. The derivative is then defined as the slope of this tangent line. In our opinion, this is the wrong approach. The first impression is important, the first impressions lasts, so the way our students think about the derivative depends on the introduction. The slope of a tangent line is a statical concept, but the derivative should be a dynamical concept. Calculus is the mathematics of change, and the most important meaning of the derivative is that it is a measure of change: the rate of change. From a semantic and didactic point of view, there is a big difference whether, in the mind of the students, the word derivative conjures up first the rate of change rather than the slope of a tangent line. Furthermore, our starting point is, that every concept should be developed in the context of a scientific question. Our introduction starts with an experiment, the cooling of hot coffee. The process of cooling will be measured, and the data can be analyzed. Thus we start with a very fundamental question: How can I find a way from data to functions, how can I find and construct models which represent the measured data and provide an insight into the underlying processes. There are many modelling approaches. The most basic approach is an empirical. The idea is to fit a curve through the given set of data and then use this curve to predict outcomes where there are no data. The mathematical method of this approach is regression. The disadvantage of this approach is that we cannot be confident that the found function applies outside the range of the available data and that the parameters have no meaning. A much more substantial and elaborate modelling process is at work with differential equations. With this approach we formulate mathematical equations, which describe the basic fundamental relationships between the variables of the problem and their rates of change. In other words we formulate differential equations. We will show how we can introduce with this approach not only the notion of derivative but also the concept of a differential equation and a numerical method to solve this equation. When we analyze the data, it is important to focus on the rate of change of the considered physical variable, in our introduction the temperature of the coffee. Here is an example of how the graphic and numerical representation of the measured temperature looks like

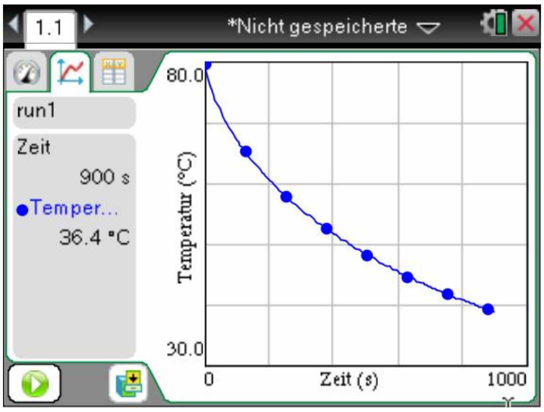
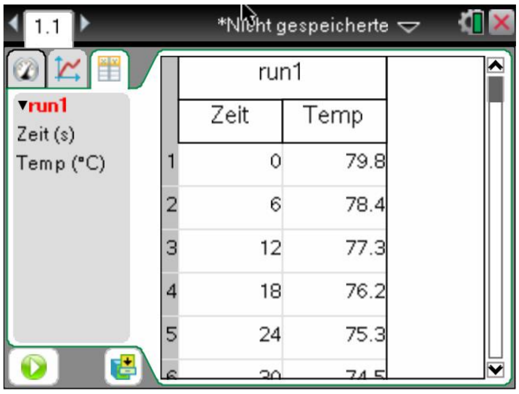


Figure 1: cooling of the coffee



	run1	
	Zeit	Temp
1	0	79.8
2	6	78.4
3	12	77.3
4	18	76.2
5	24	75.3

Figure 2: numerical data

If we focus on the rate of change of the temperature of the coffee, the discussion and investigation of the measured data lead more or less directly to the first model, namely *Newton's law of cooling:* The coffee cools off, and it cools off faster at the beginning, when the difference between the temperature of the coffee and the temperature of the room is greater. The simplest assumption we can make is that the rate of cooling is proportional to the temperature difference between the coffee and the room. We get the differential equation

$$T'(t) = -k \cdot (T(t) - T_u),$$

$T(t)$ is the temperature as a function of the elapsed time t since the start of the measurement, T_u denotes the room temperature. $T'(t)$ will be introduced as the rate of change of the temperature and this is nothing other than the expression of how fast the temperature decreases. This is easily understood and needs no deep comprehension of tangent lines or limits although the very important concept of limit must be considered later for refining the concept of derivative and making it more precise. From the given data we can in fact just calculate *average rates of change* in very small intervals, depending on the setting of the measurement (samples/second), and we take this as an approximation of the *instantaneous rates of change* which can intuitively easily be seen and understood. It is possible to discuss at this point the difference between discrete and continuous and the transition

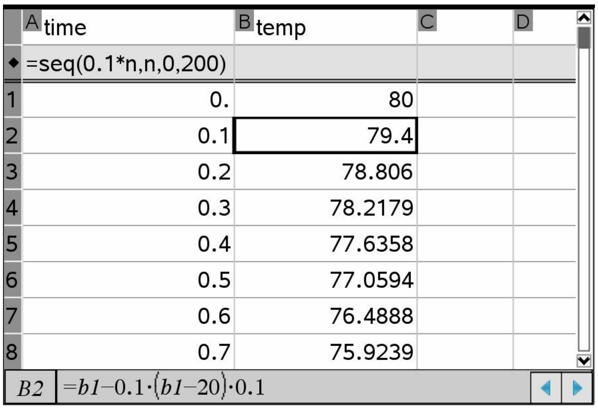
which consists in the process of making the time intervals smaller and smaller.
The starting point of Calculus is now given in form of a simple, intuitively clear model with a differential equation. From the beginning the students deal with the core concepts of modeling, rate of change (derivative) and differential equation, concepts which will successively be refined and developed in the subsequent chapters. This happens in the clear and easily understandable context of the process of cooling of a cup of coffee. To solve this differential equation we use a simple numerical method, namely Euler's Method.

4 Euler's Method, a fundamental tool

Euler's Method is known as a simple numerical method for finding a solution of an initial value problem. An initial value problem consists of a differential equation and an initial value or a set of differential equations together with initial values. Because we have now more elaborate and more efficient methods like Runge-Kutta, Euler's Method has fallen somewhat into oblivion. However Euler's Method is of great didactic value and is a fundamental tool for teaching important concepts of calculus. Its simplicity allows us to direct the attention to the central aspects instead of struggling with technical difficulties. Let us see how Euler's Method works in the previous example, using TI-Nspire. There are several possibilities, each one with its own didactic and mathematical value.
Example: $T'(t) = -k \cdot (T(t) - T_u),\ k = -0.1,\ T_u = 20°\text{C}$, initial value $T(0) = 80°\text{C}$.

Lists & Spreadsheet

- If we choose $\Delta t = 0.1\,\text{min}$ and a period of $20\,\text{min}$ we define in the first column the time t as a sequence:(=seq(0.1*n,n,0,200)).
- In the first cell of the second column (B1) we write the initial value (80).
- In the second cell (B2) we calculate this value with Euler's Method: (=b1-0.1*(b1-20)*0.1) using the fundamental recursion $T(t+\Delta t) = T(t) + T'(t) \cdot \Delta t$.
- With the command "fill" we copy the formula of B2 to the whole column.
- We denote the columns A and B with "time" and "temp" respectively and in a Graphs & Geometry-page we can plot the graph as a scatter-plot.



	A time	B temp	C	D
◆	=seq(0.1*n,n,0,200)			
1	0.	80		
2	0.1	79.4		
3	0.2	78.806		
4	0.3	78.2179		
5	0.4	77.6358		
6	0.5	77.0594		
7	0.6	76.4888		
8	0.7	75.9239		

Figure 3: Euler's Method

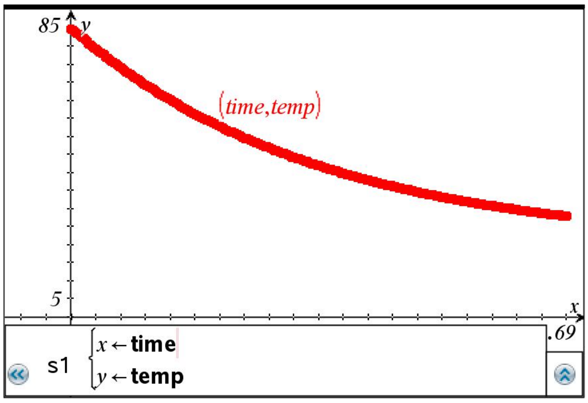


Figure 4: scatter-plot

With sliders it is possible to change the parameters $T(0)$, T_u and k dynamically.

Calculator, user-defined function

With TI-Nspire[TM] CAS it is possible to define a function. This function can then be used in all applications.

- In the application "Calculator" choose [menu, 9: Functions&Programs, 1:Program Editor, 1:New]
- Choose a name, e.g. "cooling" and choose the type: Function
- The function can now be defined as follows. In this example $dt = \Delta t$

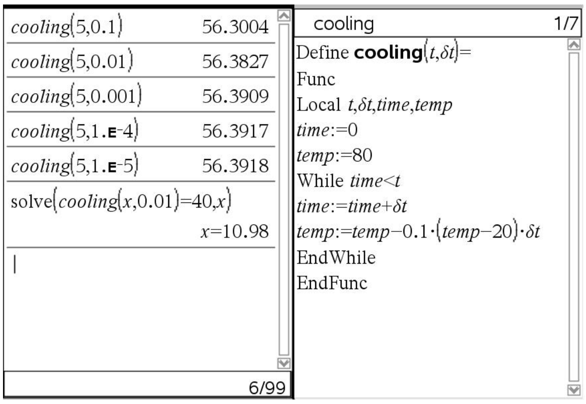


Figure 5: Userdefined function

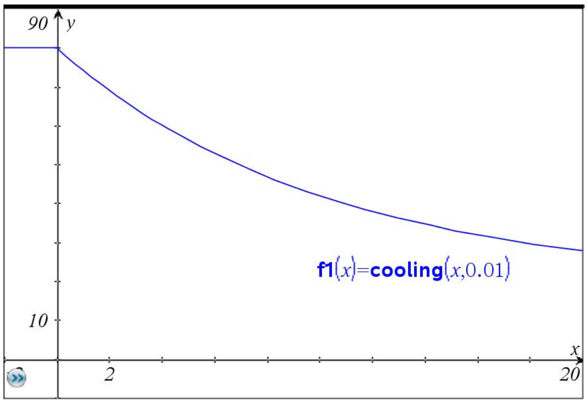


Figure 6: plot

Graphs, Graph-Type: Differential Equations

Since the version 3.0 it is possible to graph the solutions of differential equations directly with TI-Nspire[TM].

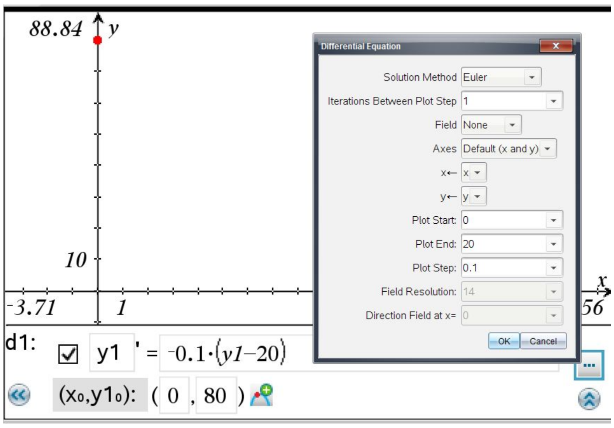


Figure 7: Graph Type:DE

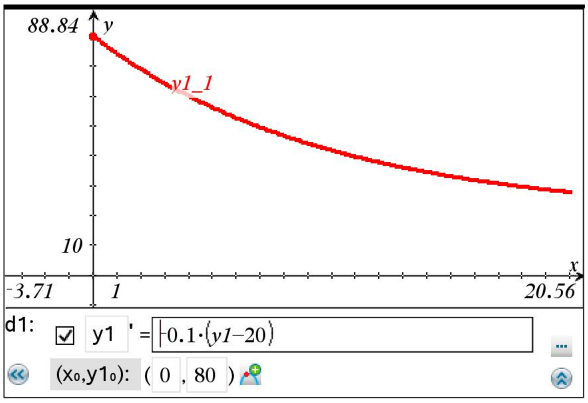


Figure 8: plot

Euler's Method makes it possible to construct the function $t \longmapsto T(t)$ and it provides the insight that an initial-value-problem has a unique solution, a very important theorem, the theorem of Picard-Lindelöf.

Defining Euler's Number e

The solution of the initial-value-problem $y' = y,\ y(0) = 1$ is the exponential-function $y = exp(x) = \mathrm{e}^x$. If you solve this problem with Euler's Method for $y(1) = e$ you get the definition

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

:

x	y	y'	$y' \cdot \Delta x$
0	1	1	$1 \cdot \frac{1}{n}$
$\frac{1}{n}$	$1 + \frac{1}{n}$	$1 + \frac{1}{n}$	$(1 + \frac{1}{n}) \cdot \frac{1}{n}$
$\frac{2}{n}$	$(1 + \frac{1}{n})^2$	$(1 + \frac{1}{n})^2$	$(1 + \frac{1}{n})^2 \cdot \frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{n}{n} = 1$	$(1 + \frac{1}{n})^n$		

Fundamental Theorem of Calculus

Euler's Method also makes it possible to proof the Fundamental Theorem of Calculus by solving the initial-value-problem $y' = f(x),\ y(a) = 0$. The exact solution is, of course, $y = F(x) - F(a)$, particularly $y(b) = F(b) - F(a)$. If you solve this initial-value-problem with Euler's Method you get a Riemann sum of f over the Interval $[a; b]$ and therefore for $\Delta x \to 0$ the integral $\int_b^a f(x)dx$ and hence

$$\int_b^a f(x)dx = F(b) - F(a),$$

details are shown in [2]

5 Mathematical modelling with differential equations

In our everyday lives, models have become an important part. They range from local personal decisions -do I bring along my raingear for tomorrow's hike- to global decisions which have a profound impact on our future. The main strengths and objectives of models are on the one hand the provision of a deeper understanding of the involved processes and systems, and on the other hand the predictive power which helps much in decision-making. Mathematical models are used extensively in ecology and biology to examine e.g. population fluctuations, the spread of a disease, erosion, the spread of pollutants etc. In fluid mechanics, models are used to design racing yachts or to understand and predict the formation and development of tsunamis. There are many different modelling approaches such as empirical models, simulation models, stochastic models, deterministic models. The last approach is widely used, and this is the one which works with differential equations. We use this approach which contains two fundamental ideas: The first is that we can write down equations for the rates of change of the relevant variables of the model. These differential equations reflect the important features of the process we seek to model. The second idea is, that these equations determine the variables as functions of time which enables us to make predictions.
In sum: **differential equations define functions.**
We begin with four simple and fundamental models of a single population: linear, exponential, limited and logistic. The new point of view is that these growing processes are now defined with differential equations. Thus we define the linear function with the equation $y' = m$, (m constant), and the exponential function with the equation $y' = k \cdot y$. These functions are growing without any limit with the time. For this reason we introduce the concept of *saturation* or *carrying capacity* S and limiting respectively braking factor. When the dependent variable $y = f(t)$ of the function approaches S then their rate of change will be successively slowed down and reach zero when $y = S$. This can be done with the factor $b = (1 - \frac{y}{S})$ (braking factor). This factor is $b \approx 1$ when $y << S$ and $b \approx 0$ when y is near S. Thus we define the limited growth and the logistic growth with the differential equations $y' = (1 - \frac{y}{S})m$ resp. $y' = (1 - \frac{y}{S})ky$.
After the single population models we continue with interacting population models. These models are relevant where two or more populations depend on each other. The most important examples are predator-prey interaction, competing species interaction and symbiosis. We will have a closer look at some of these systems in the next section.
We also study a model with three populations, the famous S-I-R-model. Let us have a

closer look at this model (a more detailed description can be found in [2] and [5]).

The S-I-R-model is used to describe the spread of a contagious disease. The population is divided into three groups:

- **Susceptible**: those who have never had the illness and can catch it;
- **Infected**: those who currently have the illness and are contagious;
- **Recovered**: those who have already had the illness and are now immune.

Model Assumptions:

- Random differences between individuals can be neglected.
- The disease is mild, so anyone who falls ill eventually recovers.
- All those who recover from the disease are then immune.
- We neglect the latent period for the disease, setting it equal to zero.

Compartment or flow diagram

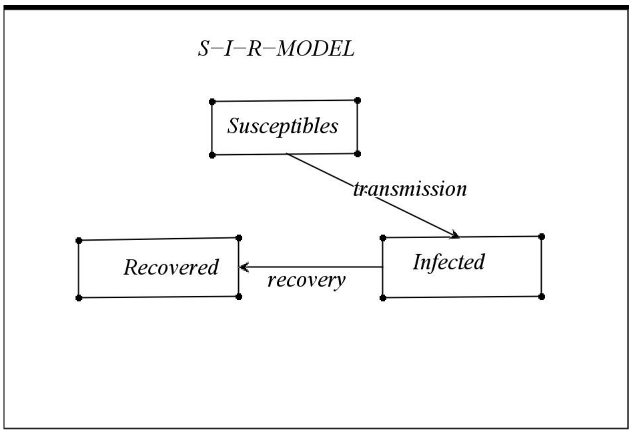


Figure 9: input-output diagram for the S-I-R-model

- **The rate of transmission**: Consider a single susceptible individual during a small interval Δt. Let's p denote the probability that this person will be infected when he meets an infected person. The probability that this person meets an infected person during an interval of length Δt proportional to the number I of infected and to the length of the considered interval, thus $q \cdot I \cdot \Delta t$. Therefore during the interval Δt, each susceptible individual will be infected with the probability $p \cdot q \cdot I \cdot \Delta t = a \cdot I \cdot \Delta t$. The parameter $a = p \cdot q$ is called *transmission coefficient*. The number of newly infected individuals during Δt is therefore $a \cdot S \cdot I \cdot \Delta t$ and the rate equation for S is:

$$S' = -aSI$$

The value of a depends on the level of contagiousness of the disease, but also on the general health of the population and the level of social interaction between its members. One strategy for dealing with an epidemic is to alter he value of a. Quarantine, for instance, will do this.

- **The rate of recovery**: Assuming that the rate of infected individuals who recover is directly proportional to the number I we write

  $$R' = b \cdot I$$

  The parameter b is called *recovery rate*, its reciprocal b^{-1} can be identified with the residence time in the compartment I, in other words b^{-1} is the average duration of the illness.

The complete model now has the form

$$\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \end{aligned}$$

Combines with initial values these equations define three functions $S = S(t)$, $I = I(t)$ and $R = R(t)$. We can solve these equations with a numerical method (Euler's method or Runge-Kutta methods). With TI-Nspire it is possible to investigate the influence of the parameters a and b on the behavior of the system.

Example

This example was given by Murray, based on data compiled by the British Communicable Disease Surveillance Centre (British Medical Journal, March 4 1978, p. 587)[4]. The event was a flu epidemic in a boys boarding school in the north of England. There were 763 resident boys, including one initial infective. The data for the two-week epidemic are given in the list below, which was constructed by reading values from the graph in the original publication. The values of the transmission rate and the recovery rate can be found with sliders. The data appear to agree well with the model's predictions for $a = 0.00218$ and $b = 0.44$

day	inf
0	1
1	3
2	7
3	25
4	72
5	222
6	282
7	256
8	233
9	189
10	123
11	70
12	25
13	11
14	4

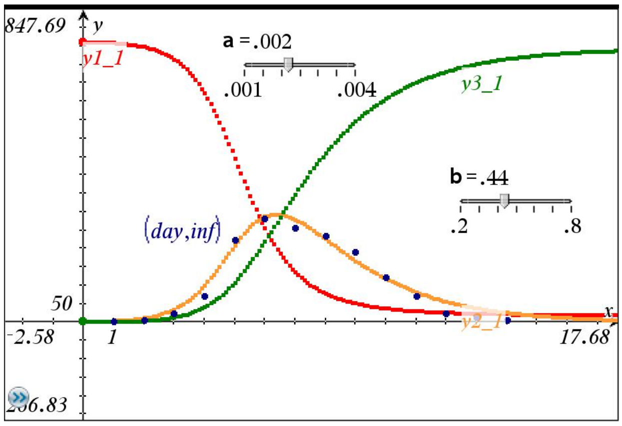


Figure 10: Numerical solution of the differential equations for an influenza epidemic in an English boarding school in 1978. The black dots correspondent to the original data.

There are a number of extensions of the basic S-I-R-model, for instance with the incorporation of a latent period or continuous vaccination. More complicated models involve populations structured into several groups, such as age groups or social groups.

6 Dynamical Systems

The modelling of the behavior of physical systems as they evolve over time often takes the following form: We have two (or more) variable quantities $y1$ and $y2$ that are functions of time, and we want to find the nature of these functions. Empirical knowledge and assumptions (e.g. physical chemical or biological laws) enable us to formulate how these functions $y1(t)$ and $y2(t)$ are changing with time. Therefore we can calculate $y1'(t)$ and $y2'(t)$ whenever we know the values of $y1$ and $y2$ and possibly t.

In other words, we have a system of differential equations. Such a set of differential equations is called a **dynamical system**. Dynamical systems play important roles in all branches of science.

In many instances, the rules determining $y1'(t)$ and $y2'(t)$ depend only on the current values of $y1$ and $y2$, but not on the value of t. In this case, the knowledge of the current state of the system is sufficient for determining the future and past states of the system. This kind of deterministic systems are called **autonomous systems**.

Thus an autonomous dynamical system consists of an abstract phase space or state space, whose coordinates describe the state at any instant, and a dynamical rule that specifies the immediate future of all state variables, given only the present values of those same state variables.

Dynamical systems are deterministic if there is a unique consequent to every state, or stochastic or random if there is a probability distribution of possible consequents (the idealized coin toss has two consequents with equal probability for each initial state). We do not consider stochastic systems here.

One of the most important techniques for studying the behavior of autonomous systems is the **Phase Space Analysis** or **Phase Plane Analysis** in the two dimensional case. This is especially important for nonlinear systems, since there is usually no analytical solution for a nonlinear system. In the previous chapters we solved these differential equations numerically:

From a given starting point we used a numerical method like Euler's method to get values for $y1$ and $y2$, this means solving an initial value problem. Each starting point generally leads to a different solution. In autonomous systems there is another very powerful way of visualizing the solutions. This way enables an overview over the entire system, over all possible solutions and the long time behavior. Moreover, the effect of different parameters on the behavior of the system can be investigated.

A short description of the method of Phase Plane Analysis

Phase plane analysis is much easier to explain with an example than in any other way. We will consider the Lotka-Volterra -model, which is the basic model for predator-prey interactions. This was one of the first mathematical population models (1924/25), proposed as a way of understanding why the harvest of certain species of fish in the Adriatic Sea exhibited cyclical behavior over the years.

Let us consider two species, rabbits and foxes in an environment. Let $y1$ denote the number of rabbits and $y2$ the number of foxes, the time t will be measured in months. The most basic model as proposed independently by Lotka and Volterra looks as follows:

The model is based on the following assumptions:

- In the absence of foxes, the rabbit population grows exponentially.
- The population of rabbits declines at a rate proportional to the number of encounters between rabbits and foxes and this number is proportional to the product $y1 \cdot y2$.
- The birth rate of the fox population is proportional to the number of foxes and to the number of rabbits.
- The foxes die off at a rate proportional to the number of foxes present.

These assumptions translate into these differential equations:

$$\begin{aligned} y1' &= a \cdot y1 - b \cdot y1 \cdot y2 \\ y2' &= c \cdot y1 \cdot y2 - d \cdot y2 \end{aligned}$$

For the sake of illustration we take for example: $a = 0.1,\ b = 0.005,\ c = 0.00004,\ d = 0.04$

- **State, phase space**
  Instead of plotting the values of $y1$ and $y2$ against the time t we take these values as coordinates of a point in the $y1$-$y2$-plane. As the system changes, the point $(y1, y2)$ will trace out a curve in this plane. The ordered pair of numbers $(y1, y2)$ is called a *state* and the portion of the plane corresponding to physically possible states is called *phase space* or *phase plane* respectively.

- **trajectories** The curve that an arbitrary starting or initial point $yi1, yi1$ traces out as time is going on is called *trajectory*

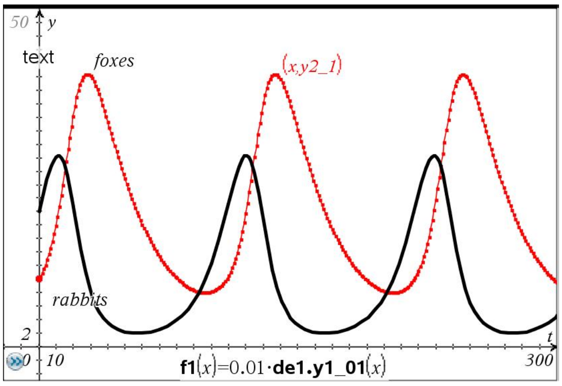


Figure 11: time-graph of $0.01y1$ and $y2$

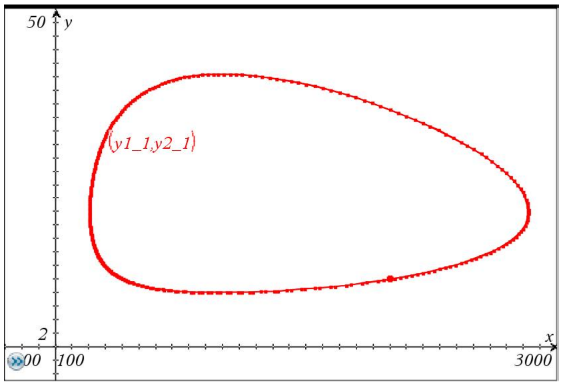


Figure 12: trajectory

- **nullclines**
  $y1' = 0$ implies that either $y1{=}0$ or $y2 = \frac{a}{b} = 20$ and
  $y' = 0$ implies that either $y2{=}0$ or $y1 = \frac{d}{c} = 1000$
  The set of points in the phase plane for which $y1' = 0$ resp. $y2' = 0$ are called *nullclines*, more precisely $y1$-nullclines resp.$y2$-nullclines.

- **steady state or equilibrium point**
  Each intersection-point between a $y1'$-nullcline and a $y2'$-nullcline is a steady state or equilibrium point. These equilibrium points can be stable, unstable or neutral.

- **vector field**
  We can assign to each point in the state space a *vector* $\vec{v} = \begin{pmatrix} y1' \\ y2' \end{pmatrix}$. This vector points in the direction of the trajectory through this point therefore all trajectories are at every point tangential to the vector at this point.

- **isosectors**
  The spaces in the phase plane that are bordered by the nullclines are called *isosectors*. In each isosector $y1'$ is always positive or always negative, the same is true for $y2'$.

Therefore in each isosector there is a qualitative direction of the trajectory, namely to the right right ($y1' > 0$) or to the left ($y2' < 0$) and up ($y2' > 0$) or down ($y2' < 0$).

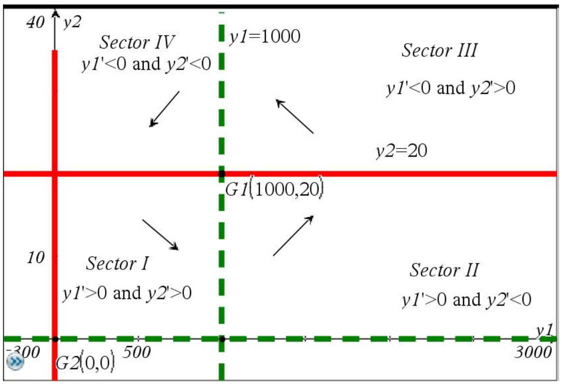


Figure 13: nullclines and isosectors

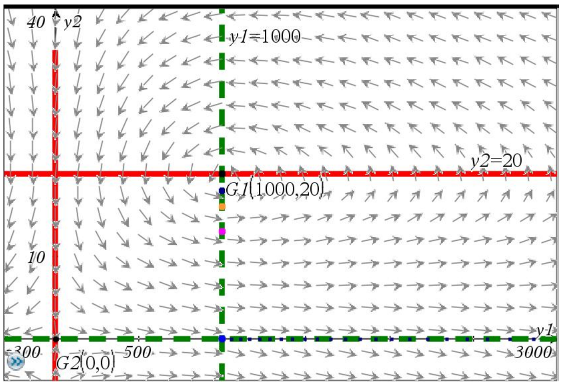


Figure 14: direction flield

For the classical Lotka-Volterra model it can be proven that all trajectories are closed, so the system behaves periodically for all values of the parameters [a]. In the workshop you will meet other predator-prey models.

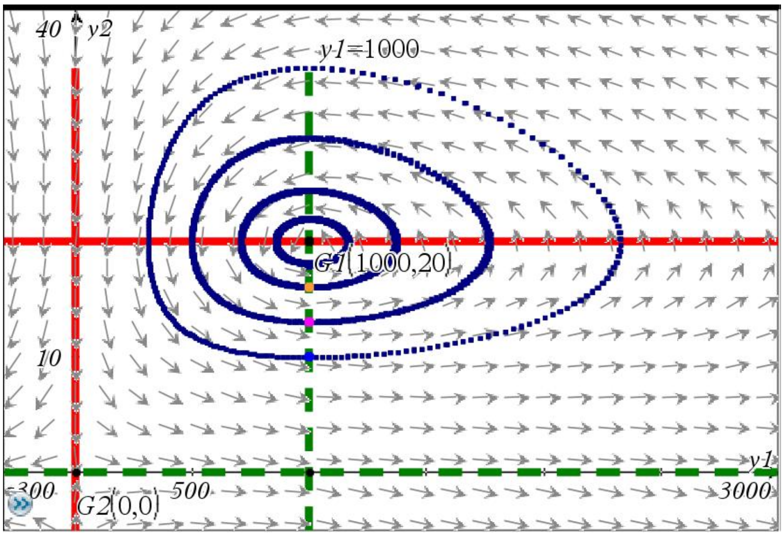


Figure 15: phase portrait

Let us consider another system. This system plays a role in the investigation of neurons and is called *FitzHugh-Nagumo model* [6]. A neuron can be stimulated with an input, such as an electrical current I. After the stimulus, the neuron is excited. The state of this excitation is described by the variable $y1$, which represents the voltage in the neuron as a function of time. When a neuron is excited, physiological processes in the neuron will cause the neuron to recover from the excitation. The variable $y2$ represents this recovery.

The model can be described with the following equations:

$$y1' = y1 - \frac{1}{3}y1^3 - y2 + I$$
$$y2' = 0.1 \cdot (b + c \cdot y1 - y2)$$

The $y1$-nullcline is $y2 = -\frac{1}{3}y1^3 + y1 + a$. The maximal slope of the $y2$-nullcline is $\frac{dy2}{dy1} = 1$ at $y1 = 0$. The influence of the stimulus I is just a vertical shift.
The $y2$-nullcline is the straight line $y2 = c \cdot y1 + b$.
For $c > 1$ there is always exactly one intersection regardless of the value of b.

With no stimulus $I = 0$ there is a stable equilibrium and thus a constant voltage, the same is true if the stimulus is weak.

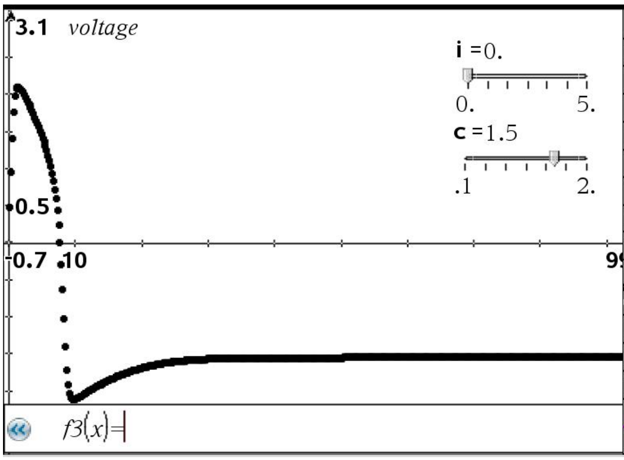


Figure 16: Stimulus $I = 0$

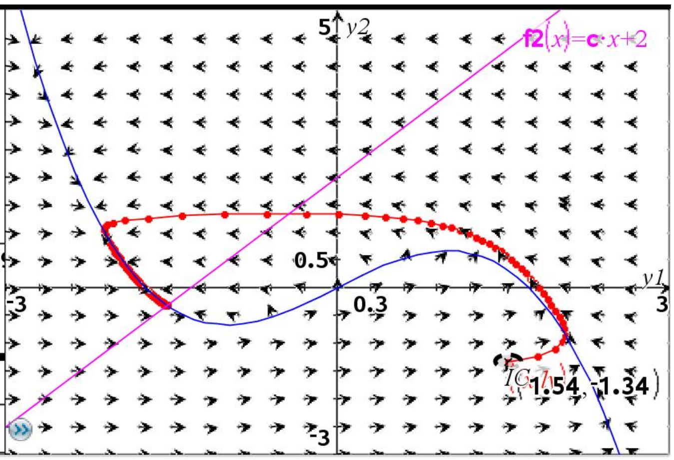


Figure 17: stable equlibrium

A greater stimulus leads to a so called *limit cycle*, the voltage is oscillating, we have a firing neuron.

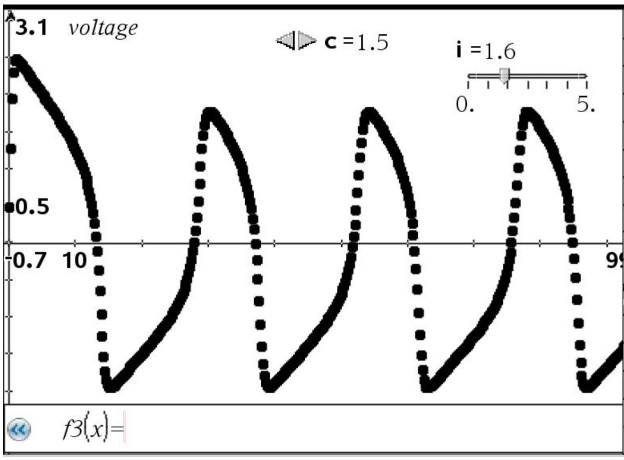


Figure 18: Stimulus $I = 1.6$

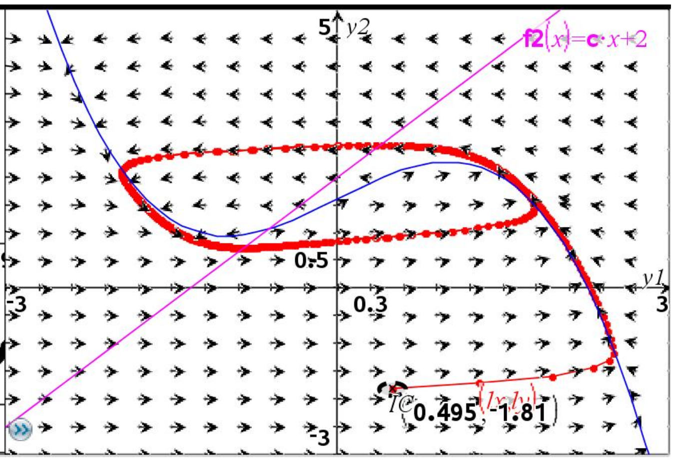


Figure 19: limit cycle

If the stimulus is beyond a certain level, again a stable equilibrium exists and the voltage is no more oscillating.

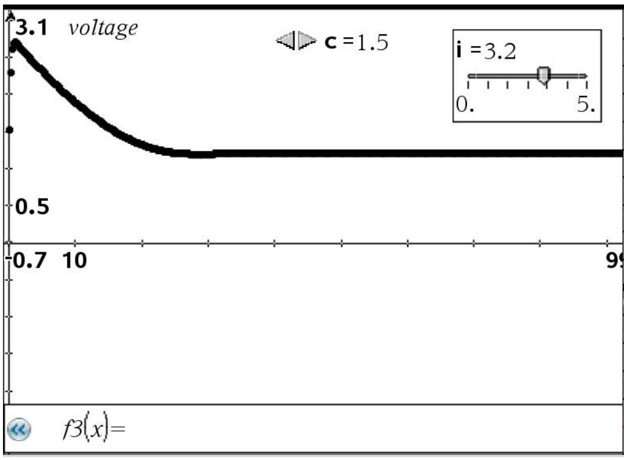


Figure 20: Stimulus $I = 3.2$

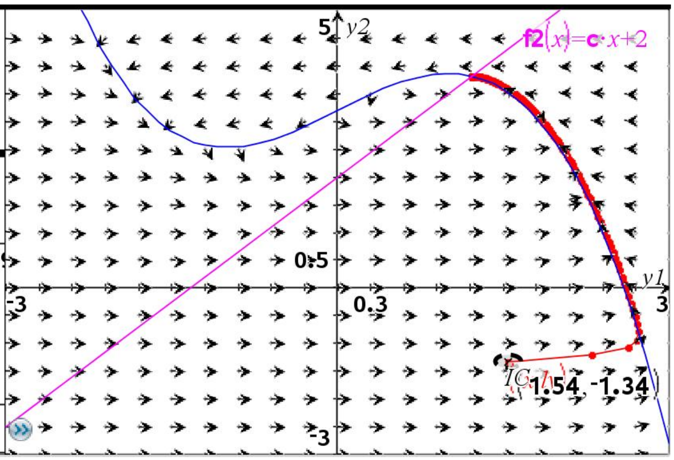


Figure 21: stable equlibrium

This example shows how the change of a parameter can dramatically change the behavior of the system.

7 Numerical graphical and algebraic methods

Numerical methods have been for a long time the last recourse to be used when no clever techniques for producing a closed form solution to the problem could be found. Students traditionally expected that most problems would be tractable with appropriate analytical techniques, and most of the books contained almost exclusively this kind of problems. With the broad availability of technology the position is now reversed. Students can be taught to approach every differential equation or every problem of integration, for instance, knowing in advance that it will be solvable by numerical methods at least, and in some lucky cases, an analytical solution is possible.

The algebraic viewpoint is central in a traditional calculus course. But this kind of course is isolated from the disciplines it serves. Technical, contextfree algebraic manipulations do not occur in biology, chemistry, economics and even physics. Algebra was widely used in calculus for the last three hundred years because numerical and a lot of graphical methods were practically inaccessible before the invention of the computer. Technology now allows direct numerical and graphical solutions to real problems. Therefore, nonalgebraic methods must play a central role in a contemporary calculus course. This shift in attitude is very important as it enables us to use the universal concepts of calculus much more effectively and these concepts are seen in a more universal light.

8 Conclusion

We have tried in this paper to show how important differential equations and dynamical systems are and that these topics are central in most of the applications of calculus. Thanks to the technology and with the emphasis on numerical and graphical methods it is possible to approach these topics at an introductory level. The ideas and concepts of calculus continue to be in the center, rather than the numerical and graphical methods. These methods merely serve as tools for dealing with these ideas and concepts.

Our course requires about 160 lessons which is about the same as a traditional course. With the help of a TI-Nspire-CAS the students are able to solve the usual examination problems of a traditional calculus course because TI-Nspire-CAS is also good and easy to use for symbolic algebraic manipulations. But beyond that the students will know how to solve the much more important real problems of modelling, differential equations and dynamical systems.

9 Bibliography

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