Ostende 2013

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Workshop „Difference Equations"

Part 3: Conjectures about adaptions of Heron's method for $\sqrt[N]{A}$

The Austrian Mathematical Society publishes so called "Math letters" (in German "Mathe-Brief") with materials for teachers dealing with several mathematical topics. This part of the workshop is a technology supported approach to the topic of Mathe-Brief Nr. 33 published by Fritz Schwaiger entitled "Babylonisches Wurzelziehen" (http://www.oemg.ac.at/Mathe-Brief/mbrief33.pdf)

Heron's method is an iterative method of approximating $\sqrt{A}$:

For A>1 and u_0=A the sequence defined by $u_{n+1} = \frac{1}{2}\cdot\left(u_n + \frac{A}{u_n}\right)$ is convergent and the limit is $\sqrt{A}$.

Conjecture: Is $u_{n+1} = \frac{1}{2}\cdot\left(u_n + \frac{A}{u_n^{N-1}}\right)$ a suitable iterative method for approximating $\sqrt[N]{A}$?

Take A = 5 and investigate the conjecture for n = 3, 4, 5:

(1) Approximating $\sqrt[3]{5}$: $u_{n+1} = \frac{1}{2}\cdot\left(u_n + \frac{5}{u_n^2}\right)$

(2) Approximating $\sqrt[4]{5}$: $u_{n+1} = \frac{1}{2}\cdot\left(u_n + \frac{5}{u_n^3}\right)$

(3) Approximating $\sqrt[5]{5}$: $u_{n+1} = \frac{1}{2}\cdot\left(u_n + \frac{5}{u_n^4}\right)$

The investigation should proceed in two phases:

Phase 1 (experimental phase): Draw the graphs of the sequence in the "time mode" and the "web mode". Is a convergence observable?

Phase 2 (exactifying phase): Calculate the fixed points and investigate the character of the fixed point by using the fixed point theorem.

Addition:

The fixed point theorem

A fixed point x* of a difference equation $x_n = f(x_{n-1})$ *(f is continuous and differentiable) is an attractive fixed point, if* $\left|f'(x^*)\right| > 1$and is distractive if $\left|f'(x^*)\right| > 1$