Ostende 2013

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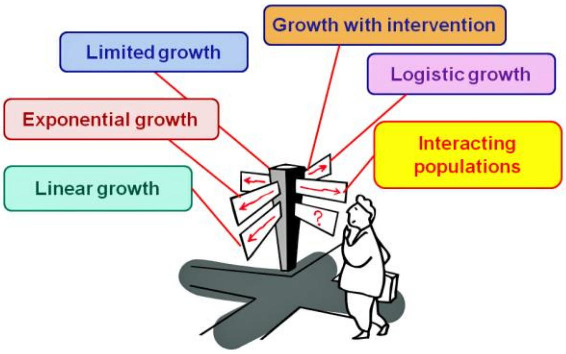
Workshop „Difference Equations“

Part 1:

Some sorts of specific growth processes which can be described by difference equations



(1) Exponential growth

Real model	Mathematical model
Characteristcs: – The rate of change is proportional to the actual stock. The increase is not constant – The same time period belongs to the same growth factor. **“Word-formula”** *“New population = old population + increase”* *The increase is proportional to the actual stock*	**Difference equations** ➢ y(n) - y(n-1) = r.y(n-1) growth rate r (per step), Starting value y(0) ➢ y(n) = y(n-1) + r.y(n-1) ➢ y(n) = y(n-1).(1+r) growth factor q = (1+r); starting value y(0) ➢ y(n) = q.y(n-1)

(2) Logistic growth

Real model version 1:	Mathematical model
Characteristcs: – Growth depending on the value of the actual stock and the free space. – The relative change is decreasing with a growing number of individuals. **“Word-formula”** *“New population= old population + increase”* *The increase is proportional to the actual stock and free space.*	**Version 1** **Difference equations** y(n) - y(n-1) = r.y(n).(G - y(n-1)) growth rate r, growth limit (capacity limit) G, starting value y(0) y(n) = y(n-1) + r.y(n).(G - y(n-1))

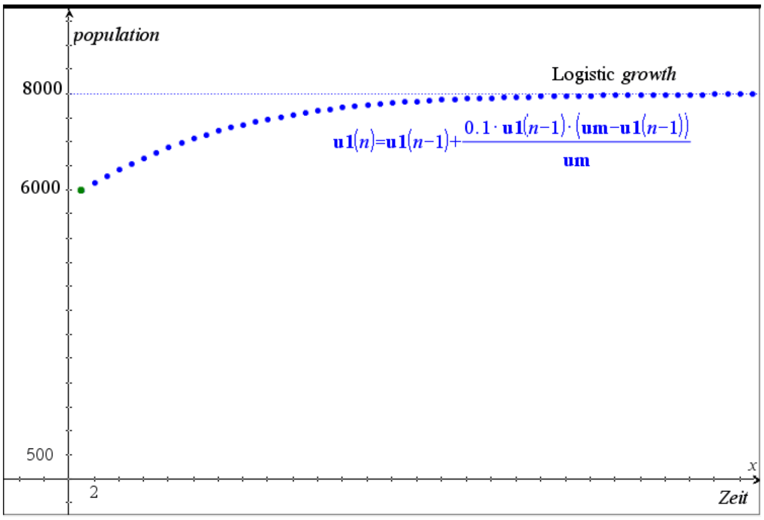
Real model version 2	**Version 2:**
Characteristcs: – Growth depending on the value of the actual population and the free space. – The relative change is decreasing with a growing number of individuals **"Word-formula"** *"New population= old population + increase"* *The increase is proportional to the actual population and the relative change of the free space.*	**Difference equations** $y(n) - y(n-1) = r \cdot y(n-1) \cdot \frac{(G - y(n-1))}{G}$ $y(n) = y(n-1) + r \cdot y(n-1) \cdot \frac{(G - y(n-1))}{G}$ growth rate r, growth limit (capacity limit) G, starting value y(0)

The differences between the two versions are the several parameters: r (in version 1) versus r/G in version 2). The parameters usually are found by experimental investigations.

Example 6: A fish population

Currently about 6000 fish are living in a lake (u0). The maximum amount of the fish population is estimated with about 8000. The growth rate of the fish population is estimated with10%.

Develop a mathematical model of the evolution of the fish population including a graphic representation of the growth process.



(3) Limited growth

Real model	**Mathematical model**
Characteristcs: – The rate of change is proportional to the available free space (e.g. living space for biological populations). The increase is not constant **"Word-formula"** *"New population = old population + increase"* *The increase is proportional to the available free space.*	**Difference equations** ➢ y(n) - y(n-1) = r.(G - y(n-1)) growth rate r, growth limit G, starting value y(0) ➢ y(n) = y(n-1) + r.(G - y(n-1))

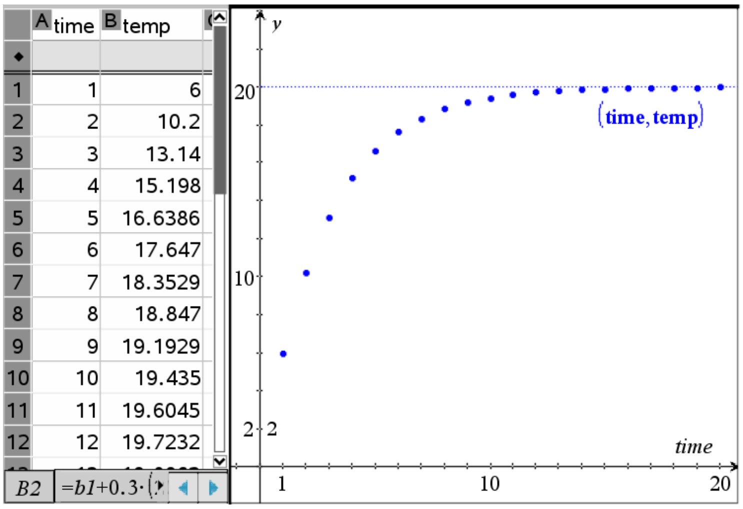
Example 7: A warming process

Food in a refrigerator has a temperature of 6°C. It is warmed to the room temperature of 20°C. The growth of the temperature is 30% of the difference in temperature between the room temperature and the current temperature (at the beginning of the current minute).

Simulate the warming process including a graphic representation

(1) with the tool “list&spreadsheet”

(2) withe a difference equation defined in the graphic window



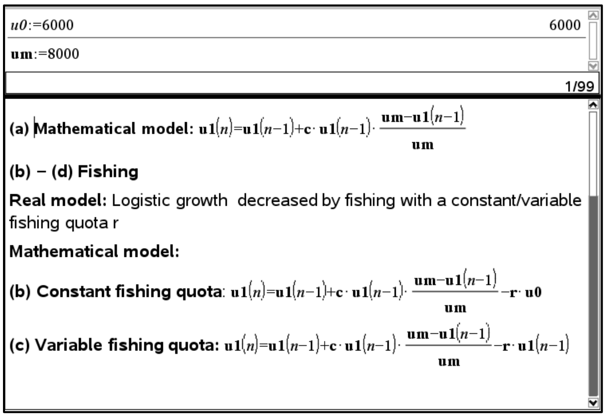
(4) Growth with intervention

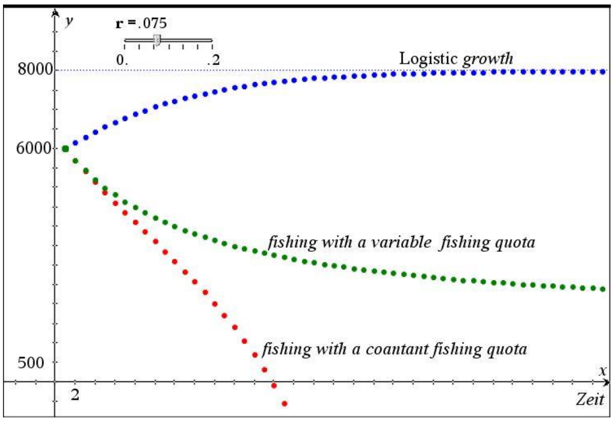
Real model	**Mathematical model**
Characteristics: The population is growing exponentially and is simultaneously increased or reduced by a certain amount **“Word-formula”** *“New population = old population + increase”* The increase is proportional to the actual population and is increased or reduced by a certain value	**Difference equations** ➢ y(n) - y(n-1) = r.y(n-1) - e growth rate r (per step), reduced amount e,starting value y(0) ➢ y(n) = y(n-1) + r.y(n-1) - e ➢ y(n) = y(n-1).(1+r) – e

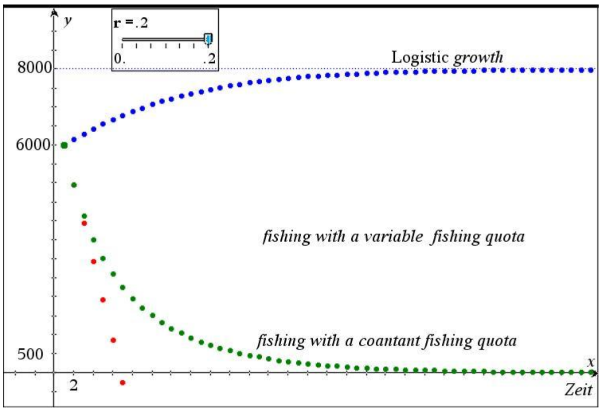
Example 8: Fishing

Currently about 6000 fishes are living in a lake (u0). The maximum amount of the fish population is estimated with about 8000. The growth rate of the fish population is estimated with10%.

a) Develop a mathematical model of the evolution of the fish population including a graphic representation of the growth process.
b) "Constant fishing quota": The agreed fishing quota is 7.5% of the population at the beginning of the process. Describe the development of the fish population by using a difference equation in the graphic window.
c) "Variable fishing quota": The agreed fishing quota is 7.5% of the current population. Describe the development of the fish population by using a difference equation in the graphic window.
d) Define a slider for the percentage rate r (0<r<0.1) and change the fishing quota starting with 7.5%. Interpret the development of the fish population for a constant and a variable fang rate.







(5) Growth of interacting populations

Often two or more populations influence each other. In the following we regard two populations B_k and Rk. Concerning the interaction between the populations we can differentiate several scenarios.

3 Models of main sorts of interacting systems:

- **Predator-prey relationship**

  The population B_k promotes the growth of R_k; on the other hand R_k impedes the growth of B_k

  $$B_{k+1} = q_1.B_k - d.R_k.B_k$$

  $$R_{k+1} = q_2.R_k + c.R_k.B_k$$

- **Competition relationship**

  Every population B_k and R_k impedes the growth of the other population.

  $$B_{k+1} = q_1.B_k - d.R_k.B_k$$

  $$R_{k+1} = q_2.R_k - c.R_k.B_k$$

- **Symbiosis**

  Every population B_k and R_k promotes the growth of the other population.

  $$B_{k+1} = q_1.B_k + d.R_k.B_k$$

  $$R_{k+1} = q_2.R_k + c.R_k.B_k$$

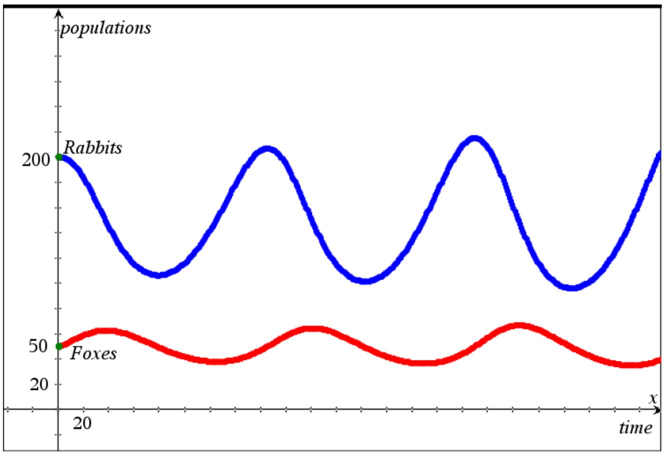
Example 9: Foxes and rabbits – a predator-prey problem

A rabbits population is annually growing with r=5%. Rabbits are hunted by foxes. The subsequent decrease of the rabbit population is proportional to the number of rabbits and foxes (kr=0.001).

The number of foxes depends on the number of rabbits. Without finding rabbits the fox population would annually decrease with s=3%. The increase of the fox population caused by rabbit hunting is proportional to the number of rabbits and foxes (kf=0.0002).

At the beginning 200 rabbits and 50 foxes would live in the district.

Describe the development of the two populations by using difference equations in the graphic window.



Example 10: HIV and the Immune System - A Mathematical Model [Lechner,1999]

- **AIDS** ⇔Acquired Immune Deficiency Syndrome
- **HIV** ⇔ human immunodeficiency virus),
- A **cytotoxic T cell** (also known as **killer T cell**) is a T lymphocyte (a type of white blood cell) that kills cells that are infected with viruses.

The terrible fact is that HI-viruses are that `successful" because their replication is susceptible to mistakes. For every mutated virus the immune system must create new specific killer cells, which can only fight this special kind. **The resistant cells act as *specialists***. On the contrary all mutating viruses can destroy *all* kinds of resistant cells against **HIV** or at least impair their function. They **work as *generalists***.

If a certain variety of viruses is exceeded, the immune system finally loses control of them and AIDS breaks out. In this way the number of virus cells increases sharply whereas the number of immune cells drastically decreases.

In school it is impossible to develop mathematical models for any numbers of mutants, and it is not so easy in general either. What we can do in mathematics education is to start with simple models of one or two mutants by using the fundamental competences about growth processes and especially about interacting systems. J. Lechner developed a program using voyage 200 for 11 mutants. Simulating needs about one hour!

The simulation of these models allows an act of reflection about the real situation.

Simulation 1:

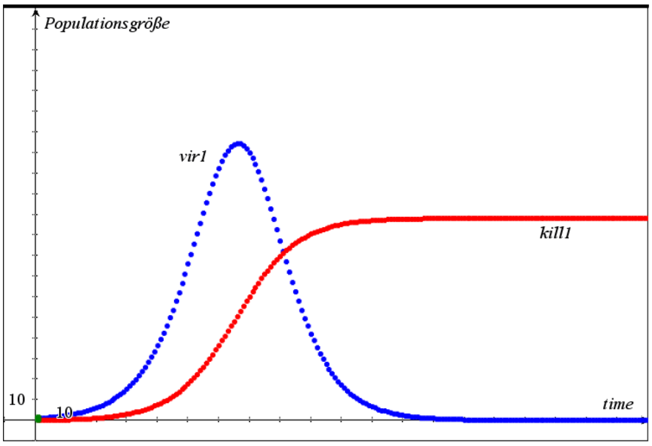
Let's start with a trivial model with only one type of virus. The difference equations are developed by using the “word-formulas” of the interacting process:

Virus 1	u1(n)=u1(n-1)+r*u1(n-1)-p*u1(n-1)*u2(n-1)	u1(0) = 1; 0≤n≤200
Killer T-cell 1	u2(n)=u2(n-1)+s*u1(n-1)-q*u1(n-1)*u2(n-1)	u2(0) = 0; 0≤n≤200

According to [Lippa, 1997, Nowak, 1992] the following parameters are realistic:

- The rate of increase of the virus. r = 0.1
- The efficiency of the immune cells in their fight of resistance. p = 0,002
- The factor of proportionality s describes the increase of the resistant cells, which is generated by the mutant of the viruses. s = 0,02
- The factor *q* characterizes the aggressiveness of the viruses. q = 0,00004

One step in time represents 0,005 years (i.e. 200 steps describe a year).

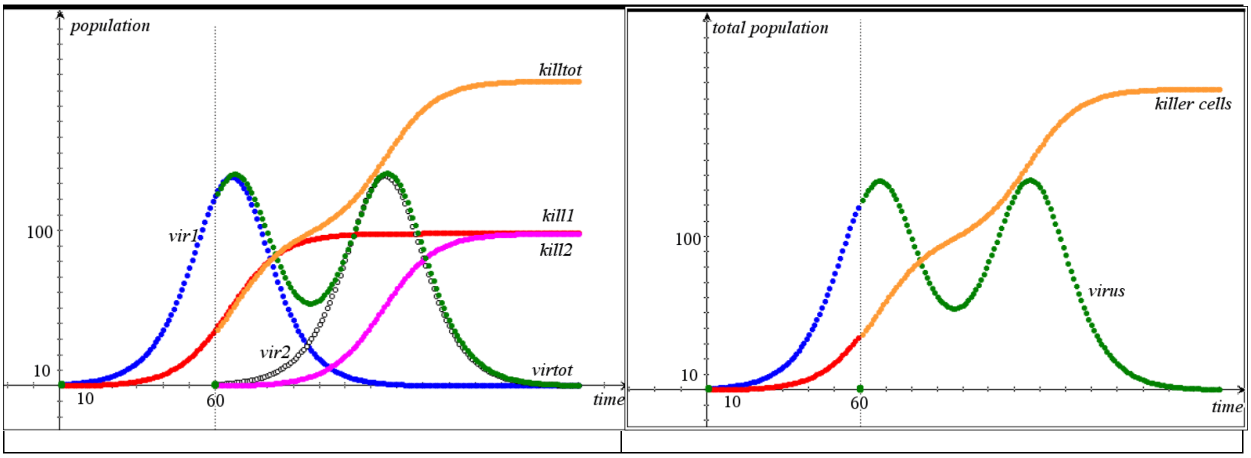


The result shows that if there exists only one mutant of the virus the resistant cells can be successful.

Simulation 2:

Now we want to work with two mutants, the second of which shall appear after 60 steps of time (which means after about 3.6 months). We have to consider, that the virus cells are generalists and therefore can fight all sorts of cytoxic T-cells while these cells are specialists and can only defeat a certain virus mutant.

Virus 1	u1(n)=u1(n-1)+r*u1(n-1)-p*u1(n-1)*u2(n-1)	u1(0) = 1; 0≤n≤200
Killer T-cell 1	u2(n)=u2(n-1)+s*u1(n-1)-q*u1(n-1)*u2(n-1)	u2(0) = 0; 0≤n≤200
Virus 2	u3(n)=u3(n-1)+r*u3(n-1)-p*u3(n-1)*u4(n-1)	u3(0) = 1; 60≤n≤200
Killer T-cell 2	u4(n)=u4(n-1)+s*u3(n-1)-q*u5(n-1)*u4(n-1)	u4(0) = 1; 60≤n≤200
Virus total	u5(n)=u1(n-1)+u3(n-1)	u5(0) = 1; 60≤n≤200
Killer T-cell total	u6(n)=u2(n-1)+u4(n-1)	u6(0) = 1; 60≤n≤200



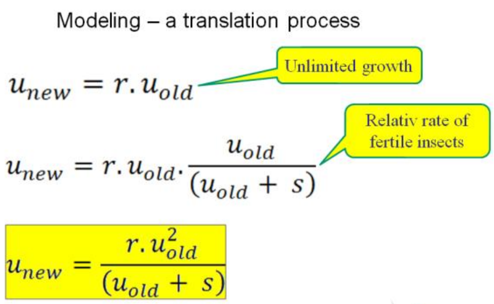
Example 11: The sterile insect technique – SIT [Timischl, 1988]

An insect population with u_0 female and u_0 male insects at the beginning may have a natural growth rate r.

To fight these insects per generation a certain number s of sterile insects is set free.

Investigate the effect of the method SIT by interpreting the growth function for several parameters u_0, r, s.

- Model assumption: r=3; s=4
- Initial values: u_0=1,9; u_0=2,2; u_0=2,0



The new population is proportional to the old population (with growth rate r) and to the relative rate of the fertile insects.

Use the “web mode” to investigate the convergence of the sequence, calculate the fixed points and determine the type of fixed point also by using the fixed point theorem.

A ***fixed point*** x* (sometimes shortened to ***fixpoint***, also known as an ***invariant point***) of a function f is a point that is mapped to itself by the function ⇔ f(x*) = x*

Is x* an ***attractive fixed point*** *of a difference equation $x_n = f(x_{n-1})$ then the sequence converges to x*:* $\lim_{n\to\infty} x_n = x^*$

The fixed point theorem

A fixed point x* of a difference equation $x_n = f(x_{n-1})$ *(f is continuous and differentiable) is an attractive fixed point, if* $|f'(x^*)| > 1$ and is distractive if $|f'(x^*)| > 1$