Is the Calculus a Must in General Education?

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Abstract

A way to teaching basic Analysis in school is presented. It has three main characteristics. We follow a genetic approach, tracing the history of the subject as an intrinsic motivation for the key ideas. We distinguish the Calculus and its syntax from the Analysis and its rather demanding semantics. The Calculus is obviously alleviated by the use of technology.

But we also use technology and scaffolding to ease insight, motivation and understanding of the principal notions of Analysis. In this context, the infinitesimal Calculus is considered an extension of elementary Algebra by rules for formal differentiation and integration. Major parts of the Calculus may be automated. This fact allows to use scaffolding and the black-box – white-box principle thanks to the available technology. We thus reduce the purely computational tasks. We allow experiments to trace the ideas behind historical discoveries. Technology lowers obstacles to acquiring ideas, insight and understanding of key notions. Moreover, we teach how to make sense out of a finite machine's output in the context of Analysis and the infinities it deals with.

The main aspects of teaching pre-univeristy Analysis are covered.

1 On Teaching and Technology

Teaching allways happens in the context of some technology. Good technology enhances teaching and bad technology creates a need for more instruction to overcome the shortcommings of the tools available.

Today's technology demonstrates that large parts of the Calculus may be automated and performed with the help of even handheld devices. This fact challenges educators and teachers of mathematics alike. Is the Calculus still a worthy subject in the context of general education? Should our syllabus be adapted to this situation? And if so, what might to be changed?

This talk reflects about the evolution of mathematics from early geometry to classical mathematics with special attention to the co-evolution of the tools available from compass and straight edge to paper and pencil, from logarithms and slide rules to calculators, work stations, smart phones or tablets. This point of view clearly suggests a genetic approach to teaching mathematics and to link the contents and methods of teaching with the technology available. In particular I propose to study some key problems that lead the way from Greek mathematics and technology to classical mathematics and present day technology. Doing so, we focus on the genesis of Algebra, the Calculus and Analysis. The insight gained from this reflection will be the basis of our approach to teaching the Calculus and the principles of Analysis using the tools available today.

It will become clear that the use of technology may naturally entail a reduction of some skills our predecessors were proud of. The formation of basic notions essential to Analysis (i.e. semantics) may be separated from the rather syntax oriented Calculus. Indeed, Calculus is an extension of basic algebra by the formal rules of the differentiation operators (i.e.

syntax). The understanding of basic notions cannot be replaced by computational skills. This understanding is necessary if we aim to teach relevant applications. Today's tools and a sound conceptual basis allow to introduce both kinds of *dynamical systems,* discrete or continuous. The discrete case is essential in many numerical procedures or in evolution models with discrete time steps. The continuous case provides the stage for models based on ODEs. The two cases are linked with one another by discretisation or the reverse process where a limit of a sequence of discrete processes ends up as a continuous one.

The following approach favors the *genetic method* over an imitation of Bourbaki style mathematics. The formalisation of mathematical content is a necessary task too. However, history shows that formalisation rather follows than preceeds experimentation, discovery, exploration and accumulation of insights and results.

An alternative approach used in higher education stresses the logical structures by definitions, theorems and proofs. This style of teaching is time saving and ends up with a lean formal theory whose meaning or possible interpretations remain largely untouched. If applied to unprepared beginners it risks to miss the point and to produce formalisms of little use.

Euclid codified greek Mathematics in his Elements only after of a long and prosperous evolution of geometry, logic, number theory. In his lifetime the decline of greek power became evident. This may indicate that formalisation also serves to concentrate and reduce a wealth of related but scattered ideas in a systematic way to essential notions and basic principles. In this way the 13 books of Euklid succeeded to preserve the legacy of Greek Mathematics. At the same time Euclid set a standard for teaching geometry to persist in Europe and elsewhere during 2 millennia. Thus Euclid became a role model for mathematicians of the formalist school. In 1899, Hilbert a leading formalist, published his *Grundlagen der Geometrie*, a modern systematic analysis of axiomatic geometry whose existence was largely due to the discovery of Non-Euclidian geometry in the 19th century.

Remark A formal definition of the *real numbers* is avoided at this stage. This task is left to university courses to serve their own intentions and aims.

However, the extension of numbers $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{R} \subset \mathbb{C}$ may be suitably explained in school in the context of algebra and by finite means.

Much Analysis and Calculus was in use for more than a century, before an attempt was made to understand the completion of the rationals $\mathbb{Q} \subset \mathbb{R}$. This step was possible only in the 19th century with major contributions by Cauchy and Dedekind. It is the essential step in creating a continuum all of whose properties are necessary to prove the major theorems of elementary Analysis. The syntax of the Calculus is a much simpler affair and is valid without this essential step. The *existence* of limits, derivatives, and primitive functions must be granted by Analysis.

2 The Origins of Analysis and the Calculus

Greek geometers used ot think in terms of ideal tools, ruler and compass. A good part of geometry was constructive, even *algorithmic* in nature and hence *finitistic* by ideology. Constructive solutions to geometric problems were restricted to finitely many fundamental constructions with ruler and compass. Another typical expression of greek mathematics is found in logics and the need for proofs. All proofs justifying certain constructions and the derivation of theorems from fundamental principles could involve only finitely many words. Moreover logics, i.e. a very disciplined use of words, had to replace the role we are used to assign to algebra, a method unavailable in Greek antiquity.

2.1 Three famous problems greek geometers tried to solve in vain

We might compare ruler and compass with the Turing machines. Then we discover a close analogy between classical problems in geometry and recent problems in algorithmic and computational mathematics. Greek mathematics left us with three famous unsolved problems:

- Given the side length of a cube. Construct the side length of another cube whose volume is twice the volume of the given cube using ruler and compass only.
- Given an arbitrary angle. How can the angle be trisected by using ruler and compass?
- Given an arbitrary circle. How can the area of the circle be represented by a square of equal area by using ruler and compass only?

None of the three problems may be solved using the prescribed tools. But this insight needed 19th century mathematics for a proof. Note that the trisection of angles as well as the solution of cubic equations would be possible by Origami constructions.

2.2 Infinity, lacking concepts, and paradoxa

Infinity transcends human imagination. Greek philosophers were aware of the problems arising when the firm ground of finitistic methods is left behind. Still some may have invented certain paradoxa of the infinite in order to demonstrate the pitfalls of naive thinking combined with talking about infinity. Zenos paradoxa [Achilles and the tortoise, the dichotomy] and the parodoxon of the flying arrow may serve to signal the dangers of naive thinking or the lack of welldefined technical terms still today. We possibly follow Zeno's intentions by recalling and commenting them before entering the territory of Analysis/Calculus.

One possibility to avoid the paradoxa is to follow the atomists whose world view is essentially finitistic and discrete. Their mathematics would reduce to combinatorics. Possibly Archimedes probed this perspective when he wrote *The Sand Reckoner*.

An alternative needs definitions and notions that avoid the paradoxa. The paradoxon of Achilles and the tortois as well as the paradoxon of the flying arrow come down to the insight that a scientific definition of velocity is connected to concepts of space and time which have to precede such discussions. The problem of the dichotomy has to be overcome before numerical methods such as bisection may be understood.

The good news is that Archimedes found a way to overcome the problems hidden in Zeno's paradoxa. This is the main point of the next section. The notion of velocity emerged only in medieval times (Oresme) but remained enclosed in academic circles. A breakthrough happend with Galileo's and finally with Newton's work in physics.

2.3 Archimedes: engineer, mathematician, genius – early roots and principles of basic Analysis.

Ruler and compass constructions allow approximate solutions up to any given positive tolerance. But this could not satisfy the requirements of rigorous philosophical standards in classical greek mathematics.

Think of an engineer designing a gear box or a mechanical orrery. In its construction wheels will have to be made with mutually prescribed proportions of the lengths of their circumferences. Greek mathematicians knew that squaring the circle and measuring the circumference of any circle are essentially the same task. A mechanical orrery of surprising sophistication made in ancient time was found near the island of Antikytera around 1900. Hence, antique craftsmen or engineers knew how to use approximations to squaring the circle of prescribed precision with success.

For the sake of preparing the essential ideas of Analysis allow me to personalise a major step of mathematical thinking. The following sketch is a kind of saga with a grain of truth rather than historical truth with a grain of salt.

About teaching In class, I might read now Ciceros 4th letter from Tusculum where he searched and found Archimedes's tomb stone and found engraved a cone, a cylinder and a sphere. Then I would discuss Archimedes's *approximations* for π which of course are the essential approximations for the content of the unit circle by regular n-gons, inscribed and circumscribed. Today this provides an exercise in programming, and it is instructive because it may show the pitfall of naive algorithms and the potential numerical degenerations due to the use of machine numbers in stead of real numbers. Here we are close to a major issue of Analysis and we dive into the problem rather than to avoid getting wet by constructing the real numbers in a 19th century fashion outside the intellectual reach of youngsters. And we may chose this approach only thanks to the easy acces to a programmable device be it a handheld, a tablet or a desktop.

The main point here is that Archimedes thought like an engineer. He overcame an ideological obstacle of greek geometry by accepting a method that lead to approximate solutions with an error that could be made smaller than any given positive tolerance. This is good enough for all practical purposes. This change of attitude is an essential step necessary for the working of what I am going to call *Archimedes's trick*.

Once we understand the approximations of the area of a unit circle up to a given arbitrary positive tolerance, we may satisfy the requirements of an engineer or craftsman who has the task to construct a mechanical device within the limitations given by his materials and his tools. In this way we are able to measure the content of any cylinder, cone or sphere for the sake of practical applications. In doing so, we use *discretisations* of an appropriate kind in order to produce results that can be proved to satisfy the precision requirements. Technically speaking we consider Riemann sums approaching a circle or half circle, a cone, a sphere, and we may do so because the algebra involved gets trivialised by *scaffolding* thanks to the CAS which simplifies the sums where necessary. Note the advantage of using algebra, a means not available to Archimedes.

Furthermore, a discussion of an ancient atomist's view may be fruitful here. For an atomist, the iterate refinements stop after a finite number of steps. In his view the area or volume reduces to a *finite* sum over the corresponding measures at the atomic level. Measuring is reduced to counting. (cf. also Archimedes's writing on the sand reckoner)

Archimedes's trick is a label to denote the method of successive approximations up to an arbitrary positive tolerance. And, moreover, the radically new convention is that a problem is considered to be *solved* (not only for all practical puroses) if this stage is reached. This suffices, of course, for all constructive approximations, and we may stay inside the rational numbers as long as we refrain from postulating the existence of limits.

In all the examples studied, the limits may be determined naively by algebraically grouping terms into a constant and terms involving the discretisation parameter. As the case may be, this parameter is n and we consider $n \to \infty$ or it is some Δx and $\Delta x \to 0$, and in the examples chosen the error terms are functions of the discretisation parameter and these functions tend to zero if the discretisation is pushed to its continuous limit.

Archimedes's trick is now applied in the educational context to explore the diverse methods for approximating the volume of a sphere (stacks of thin discs, triangulation or shelling) in analogy to similar methods for computing the area and circumference of a circle. By way of example we are lead to consider difference quotients of the area of a circle or the volume of a sphere as a function of the radius and we get into contact with the *idea* of a derivative. But beware – the name is kept in the cheek as well as 'integral' is a no-name at this stage of instruction.

Archimedes's intuition found some basic principles and essential *ideas* on the way to Analysis. For lack of algebra he was unable to contribute to what we now would call the Calculus.

2.4 Heron and iterative numerical approximation of roots

Square roots may be constructed by ruler and compass respecting the highest standards of greek philosophy. But this does not help to satsify practical needs for a good approximation in terms of fractions. Archimedes's trick does not work with a materialised tools of geometry.

The long tradition of mathematical culture in the Middle East offered an alternative. Its origin is documented in clay tablets but the name of the method is attributed to Heron, a physicist, engineer and mathematician living in Alexandria in the Hellenistic period.

Heron's method is interesting because it shows the role of numerics in antiquity and it produces in principle a series of rapidly converging rational approximations to square roots. This may be exploited to extend the role of Archimedes's trick beyond geometry to function iteration producing successive approximations to solutions that cannot be reached in finitely many steps within the given arithmetic.

About teaching In practical computation with calculators, saturation of the decimal representation occurs. Still an algebraic argument shows that errors decay in exact arithmetic. Hence Archimedes's trick applies and shows the existence of an exact square root to any positive number independently of geometry and its idealised tools.

Students' projects for an adapted version of Heron's method to arbitrary roots are an option here. They may lead to Newton's famous method being rediscovered in the context of special examples in school.

Heron in a modern perspective: Heron's iteration is a discrete dynamical system on the field of rationals $\mathbb{Q}$. Typically its fixed points lie in $\mathbb{R}$ or $\mathbb{C}$. This fact motivates the need for extensions of $\mathbb{Q}$ in the context of a smooth mathematical theory. But paradoxically all numerical computations take place in some finite subset $\mathbb{R}_{\mathbb{M}} \subset \mathbb{Q}$. Even a CAS is bound to a finite memory implying that its range of effectively computable terms remains finite all the time. Our tools do not fit the needs of the basic mathematical background in theoretical Analysis. However, the Calculus provides us with some algebraic shortcuts which open the way for solutions in finite terms under suitable conditions. The impression of an allmighty Calculus is seductive like a mirage. Numerical approximations are absolutely essential to the engineers' tricks of trade. Hence *numerical Analysis is a must in general education* complementing the applications of the Calculus in an essential way.

2.5 Dido, Pappus and Fermat: Is Nature ruled by extremal principles?

Tell about Dido and the foundation of Carthago, tell about Pappus and his school. In particular tell how Pappus explained the propagation of light rays between two points A and

B and why they ought to follow a straight line or why the angle between the incident light ray and a mirror is equal in size to the one of the outgoing light ray because of the postulated parsimony of nature.

An easy argument in Euclidian geometry suffices in order to 'prove' what Pappus claimed. But it only works under the hypothesis that shortest paths between two points are straight lines.

Tell that Pappus also claimed the bee hive to solve a minimal problem. Another claim of his says that the bee cells have hexagonal walls 'because' the bees have six legs.

An exact study reveals that bees aren't precision workers and that several different types of bee cells may be found in nature. Still T. C. Hales in 1999 published a proof that a form of the honeycomb conjecture attributed to Pappus or Varro is true in two dimensions.

Twelve centuries after Pappus, Fermat extended the principle of the shortest path for light rays and gave it a new twist. He claimed that light rays travel with finite speed depending on the medium of propagation and his principle was little more than a belief that light propagates through an optical system such as a rain drop, a lens, a prism in such a way that the time of travel along a light ray was minimal [or extremal] among all paths connecting a given starting point A to some given point B.

About teaching We take up Fermat's claim and derive the *law of Snell* in mathematics as an exercise and application of Fermat's principle. [Here we need an essential ingredient to modern Analysis: analytic geometry and its link to algebra.] Moreover we enlarge the set of exercises in nonlinear optimisation. They may be solved first using the CAS and *scaffolding* and then using the *black box – white box principle* to make acquaintance with numerical optimisation and with Fermat's forerunner for the derivative. In the case of a polynomial function p, the difference quotient $\frac{1}{h}(p(x+h)-p(x))$ may be divided out completely and then we recognize a function of x and a multiple of h. Setting $h := 0$, the function of x remains and it is what we now call the *derivative*. It is noteworthy that Fermat didn't hit the correct definition at first stroke. We use the 'error' of the famous man for didactical purposes and comment on it later on when the derivative will be defined formally.

This example is relevant since in the 20th century all physical theories were based on variational principles. This view allowed to see parallels between optics and mechanics and to free a way for quantum mechanics to amalgamate both subjects.

2.6 Kepler and the volume of barrels

When Kepler married his second wife, the wine was furnished in several barrels. But Kepler understood that the seller of the wine tried to cheat him. The volume indicated of the barrels was non-plausible to Kepler. This prompted the scientist to derive a handy formula to determine the volume of a barrel based on four measurements: The cross sections of the barrel at the bottom, the top and at mid height, and the distance between bottom and top. Keplers rule is a formula that approximates the volume of a barrel based on these data. The formula is exact, e.g., for cylinders, truncated cones, spheres and their segments or spherical zones, and in general it provides good approximations for many practical applications.

About teaching Kepler's method may be rediscovered in school. A CAS calculator and scaffolding allow to concentrate on the essential steps which consist of:

- interpolate the measures of the three cross sections quadratically.

- integrate a quadratic function over an interval using discrete approximations and idealisation by limits.

It is worthwile to extend Keplers rule and to discover *Simpson's rule.* It provides us with an opportunity for an exercise in programming an early example of a numerical procedure. This leads to first experiments in what we shall call *numerical integration* only after a systematic categorisation of all the diverse examples to be given in the sequel.

By the way, Kepler was among the first to profit from the then novel use of logarithms. Without these tables, his lifetime might not have sufficed to decypher the laws of planetary motion based on the records of the best astronomical observer of his times and his predecessor at the court of Prague, Tycho Brahe.

2.7 Aristotle and Galileo

Aristotle was an authority and part of his prestige derived from the fact that he was the teacher of Alexander the Great. Artistotle's philosophy embraced also an early form of natural science which dominated what was taught, learned and believed in European schools and universities well through the Middle Age. Aristotle claimed that a falling object accelerates in such a way that the speed is proportional to the length of the path coverd. Galileo contested this claim and was lead to think and experiment with falling bodies. His empirical approach to physics helped overcome the imposed mistaken beliefs and stagnant understanding of the natural phenomena under the authoritative regime of traditional philosophy as well as religion. Galileo dealt with models of motion under constant acceleration and hence was able to describe the trajectory of arrows or mortar balls much better than the learned philosophers or the gunners. Galileo placed mathematics at the base of understanding nature by saying that the book of Nature lays open before our eyes but it is written in geometric figures and who wants to read has to master the *language of mathematics.*

About teaching Essentially Galileo defined mean velocity as a difference quotient of positions and [mean] acceleration as a difference quotient of velocities. The case of Galileo also shows that notions are basic to understanding rather than formulae.

2.8 Pascal and Leibniz: infinitesimals and the Calculus

Pascal, a child prodigy, found about 400 theorems about projective geometry and conic sections. Among other results he showed how to construct a tangent to a conic section $\mathcal{C}$ defined by five points through one of the given points. Call it $P \in \mathcal{C}$ and *imagine* what happens in the following procedure. Step 1: construct a secant line through P and an auxiliary point $Q \in \mathcal{C}$. Step 2: move point $Q \in \mathcal{C}$ towards P and observe what happens to the secant lien. In this perspective, one may catch the seductive idea that the tangent to $\mathcal{C}$ in point P is a line connecting P with itself as the limiting case of a secant between 'immediately neighboring' points $P \approx Q$ both on $\mathcal{C}$. The idea that tangents are limiting cases of secants must have been somehow in the air. Or not? It is also known that Leibniz on a diplomatic mission visited Paris and met Pascal. So we might speculate what they talked about. A fact is that Leibniz invented a method of computing tangents for almost every curve known at his time by using the *somewhat mysterious differentials.* Leibniz's differentials were conceived as *positive but infinitely small.* His concept is suggestive but cannot be grasped easily. If 'number' means fraction there is no place for infinitesimals. Infinitesimals contradict the so called archimedian property of real numbers. A well established notion of real number was lacking at the time.

The geometric substitute, a straight line, was diffuse enough to allow speculations about the local structure of a continuum. Leibniz successfully developed a Calculus for differentials based on an extension of algebra to include operations with differentials and a pertaining convincing notation. Leibniz was focussed on syntax and he was successful and competitive with the leading mathematicians of his time. Although he was a lawyer and librarian by profession, his creative mind made him a universal genius. His character brought him into conflict with others of similar stature.

About teaching In our introduction, we paraphrase the computation of tangents in guise of *local linear approximations* following the two steps of the archimedian trick. The slope of the tangent to the graph of a function f is gotten by first computing the slope of a secant.

$$DQ(f, x, \Delta x) := \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

In the second step, we try to idealize and let $\Delta x \to 0$. Of course we discuss the problem of existence and we find that in the case of polynomials (of low degree), algebraic manipulations suffice to find the limit. By the way, this is Fermat's view on the problem.

Our first aim is to be able ot find derivatives for all polynomials. This is achieved by proving two results of the Calculus: The rules for derivatives of sums and of products. Calculus now suggests as corollaries the rules for integral powers and even fractional powers.

It is noteworthy that Pascal constructed one of the early mechanical calculators, cf. [PAS]. Did he show it to Leibniz? Leibniz propagated the idea that thinking could be formalised and hence mechanized with drastic consequences: Stop quarrelling – Let's compute!

To some extent, a CAS makes Leibniz's vision to be true. But Daniel Richardson in his 1968 thesis put an end to Leibniz's dream. Some essential questions are algorithmically undecidable in the class of elementary functions a CAS is supposed to operate on, cf. [RIC].

2.9 Newton's Principia: New Maths for new Physics

While Kepler found his laws of planetary motion by analysing empirical data, Newton made a claim about the law of gravitational attraction. However, a problem remained, the attraction exerted by masses escaped laboratory experiments in Newton's generation. The celestial bodies with their huge masses might be used to corroborate the claim. Newton's idea was to deduce Keplers laws of planetary motion as a logical consequence under the hypothesis of his universal law of gravitation. This is what Newton finally achieved. Still he had to invent new mathematical tools in order to finish the proof. Finally, Newton invented his own version of derivatives and integrals and he based mechanics on axioms and differential equations. With respect to his masterpiece, the *Philosophiae Naturalis Principia Mathematica* containing the solution of the *two body problem*, he was modest enough to remark 'I stood on the shoulder's of giants', acknowledging thus the role of many forerunners in a long tradition.

Newton's presentation however eschewed this new form of mathematics and rather relied on geometric arguments. (Was this a 'didactical' concession to his contemporaries or rather a protection for the uniqueness of his powerful invention?) Although he introduced the new discipline of *dynamical systems* with his celestial mechanics, we recognise this fact only with hindsight.

Finally Newton and Leibniz disputed bitterly over the priority of the invention of the Calculus.

About teaching While teaching Analysis to beginners, we profit from the occasion and introduce some connections between mechanics of mass points and Analysis. We consider a mass point moving along the real line. Its *position* x is a function of time $x : t \mapsto x(t)$. From any given position function we may tentatively derive a *velocity* function in two steps.

- Choose an interval $\Delta t \neq 0$ and define the mean velocity for the Interval $[t, t + \Delta t]$ by the difference quotient

$$\bar{v}(t, \Delta t) := \frac{1}{\Delta t} \cdot (x(t + \Delta t) - x(t))$$

- Remove a possible discretisation error by taking the limit $\Delta t \to 0$ and define

$$v(t) := \lim_{\Delta t \to 0} \bar{v}(t, \Delta t) \quad \text{if this limit exists}$$

Remarks

1. Note that here the derivative naturally appears as an *operator* producing functions from functions, rather than a prescription to produce numbers. This is essential for the next step where acceleration is derived from velocity by an analogous procedure.

2. Acceleration is the derivative of the velocity function:

$$a(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot v(t + \Delta t) - v(t)) \text{, if this limit exists}$$

3. Standard notations

   (a) Newton: $v = \dot{x}, \quad a = \dot{v} = \ddot{x}$
   Today, this notation is reserved for derivatives if the variable is time t.

   (b) Leibniz: $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}, \quad a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$

   (c) Cauchy: For all t in the domain of the functions:
   $v(t) = x'(t), \quad a(t) = v'(t) = x''(t)$

4. Newton's law:
   $F = m \cdot a = m \cdot \ddot{x}$ is a *differential equation* for the function $x : t \mapsto x(t)$.

5. Galieo's inertial law has the form $m \cdot \ddot{x} = 0$. This implies that either $m = 0$ or $\ddot{x} = 0$. In the second case, Galileo claims this to be equivalent to $v(t) = v_0$, a constant.

   A mathematical proof of this claim requires two arguments. The derivative of a constant v_0 vanishes as any beginner who understood the concepts will agree. The opposite conclusion, however, requires all the major theorems of elementary Analysis, possibly disguised in the Fundamental Theorem of Analysis.

6. Any free falling mass without air resistance obeys a law of the form $\ddot{x} = g = \text{constant}$ according to Galileo.

   We may check as an exercise that $\dfrac{\mathrm{d}}{\mathrm{d}t}\, g \cdot t = g$ and that $\dfrac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2} \cdot gt^2 = g \cdot t$, but for all constants x_0 and v_0 the functions $x : t \mapsto \frac{1}{2} \cdot g \cdot t^2 + v_0 \cdot t + x_0$ are solutions too. How can we see that no other solutions exist? Could mathematics possibly produce solutions that Galileo missed or, even worse, that nature forgot to display?

   Those who venture considering the vector version of this example will find the parametrisation for a parabolic flight under a constant acceleration.

3 Elementary Analysis: approximate to exact or discrete to continuous – back and forth

3.1 Discretisation

If a problem involving some continuum and hence infinitely many operations it may be reduced to a finite one by discretising. Discretisation creates some errors and loss of information. The problem is to keep discretisation errors under control. If this can be done sucessfully, a second step will follow and purge the effects of discretisation. (cf. Idealisation, below)

Examples

- *Mean values* of finitely many values sampled from some continuous function.
- *Riemann sums* for a function sampled at finitely many points.
- *Difference quotients* as the mean specific rate of change over an interval.
- *Function plotting* is always based on finitely many samples.
- *Difference equations* e.g., Euler's method, sampling a vector field

3.2 Idealisation

The errors of discretisation may often be removed by taking a suitable *limit.* Here some general properties of functions like continuity or differentiability or smoothness are usually required.

Basic Analysis defines and limits its range of validity to such cases where the two principles of discretisation and idealisation may be combined sucessfully. Counter examples may serve as a warning. However their mathematical treatment easily exceeds the possibility of a first course. Or remember the advice of D'Alembert in this context: Allez en avant et la foi vous viendra.

4 Key notions, examples, and comments

Machine numbers, rational numbers, and real numbers Every student working with a computer, even a handheld calculator, ought to be confronted with some of the unavoidable difficulties of numerical computing: roundoff, underflow, overflow as consequences of the finiteness of the set of machine numbers. Similarly, despite the famous pythagorean dogma concerning the non-existence of irrationals, the rational numbers do not suffice and the real numbers, which are needed for Analysis to work, typically have no names and pop up like random numbers from an uncountable set.

The Derivative exists if the limit of the differential quotients exists. The term 'derivative' has several interpetations.

- value of a function
- a function
- a linear operator on function spaces.

The Calculus refers mainly to the rules that hold for computing the derivative of some elementary functions based on rules for certain primitives.

The Integral comes in various forms as

- the limit of Riemann sums defines the definite integral.
- integral function $I_f : x \mapsto \int_a^x f(t)\mathrm{d}t$
- indefinite integral, the solution set of a differential equation $F' = f$ for a given continuous function f.
- a primitive function F satisfies $F' = f$ and is an element of the indefinite integral.

All these terms are connected by the statement of the Fundamental Theorem of Analysis. The definite integral of a continuous function may be defined as a limit of Riemann sums. The indefinite integral is the solution of a differential equation and the Fundamental Theorem guarantees the existence of primitive functions with continuous derivatives. Moreover it describes how to get primitive functions based on definite integrals and it parametrises the indefinite integral, i.e. all the solutions of the differential equation $F' = f$. The proof of this central result resides on all major theorems of elementary Analysis.

From the point of view of the Calculus, the role of the Fundamental Theorem rather gets belittled when reduced to its role in computing a definite integral with the help of a primitive function. Although primitive functions exist for all continuous functions given on some interval, typically primitive functions of elementary functions are not elementary any more. Here the Calculus and hence any CAS come to some unsurmountable limits. Thanks to technology, *numerical integration* with the help of a computer becomes an alternative. However this alternative typically involves sampling and discretization and some errors that cannot be removed at will by computing a limit.

Sampling and [spline] interpolation The use of graphing devices calls for some remarks about sampling. Sampling is necessary for computer graphics. It may cause artefacts that illuminate the problems of discretisation in this case. Similarly function tables involve sampling to. The reconstruction of approximate values often is done by interpolation. Interpolation is an alternative for closing gaps *independent* of the concept of limits.

Differential equations in the context of vector fields and dynamical systems are a valuable addition to basic Analysis. In view of the role ODEs played in the genesis of Analysis, some differential equations ought to belong to any introduction to Analysis. Hereby the graphing capability and *numerical simulation* might be more illuminating to a general audience than the bag of tricks offered by Calculus and of use in selected cases only.

Mean values and the Fundamental Theorem of Analysis The mean values of a continuous function may be computed either by a definite integral or by a difference quotient (involving a primitive function). Both methods lead to the same answer. This is the essence of the Fundamental Theorem of Analysis.

At the Limits of basic Analysis All cases where discretisation combined with idealisation by limits fail, are excluded from basic Analysis by definition. *Fractals* may provide examples of questions where the tools of basic Analysis do not suffice for a good answer.

Beyond the introduction It is clear that students knowing only the basic notions would be ill prepared for further studies. Some of the following subjects will be dealt with as time and students' abilities permit.

- Polynomial functions and their special properties, e.g. interpolation, numerical integration, numerical differentiation.
- rational functions
- exponential functions, logarithms and models for growth or decay
- trigonometric functions and models for vibrations
- parametrised curves and motions of mass points
- some examples for vector fields and ODEs, population models, linear ODEs with constant coefficients, vibrations.

5 Conclusion

Calculus may be automatised. This may be one reason for Calculus to be taught quite sucesssfully by drill and practice, by programming humans, so to speak. Our students deserve more general education than this.

The discussion, motivation and careful introduction of *basic ideas* and specific notions and their interrelationship is the fabric of Analysis. The operations of the Calculus may be automated and left to be performed by using a CAS. The meaning of what we are doing by using the Calculus with or without an automaton is to be found in studying Analysis. Learning Analysis must be based on gaining understanding and insight. We have a cultural heritage to pass on to the next generation – not just a finite set of rules. It is fair to answer two questions: What does it mean? – How does it work? While the working may depend on some technology, the meaning is deeply rooted in two millennia of mathematical culture. By using scaffolding and the black-box – white-box principle we may simplify the working and devote more time and efford to the meaning of what we teach.

1. *Semantics* is the essence of Analysis, a part of mathematics dealing with the continuum, real numbers, functions and limits. For the sake of general education the *Archimedian trick* based on *discretisation* combined with *idealisation* is the center piece. It explains why and how many of the applications of Analysis work. Why can the derivative be used to compute instantaneous velocity, instantaneous acceleration, the slope of a tangent, the density of a mass distribution? Why do we need integration to find the total charge from a charge density? Why is there a well defined mean for any continuous function and any finite interval of its domain? Why can we use integration for measuring curve length, areas as well as volumes? Why can we use the derivative to determine the surface of a sphere? Why can the surface of a sphere also be determined by integration? How can we find the centre of mass of a mass distribution in a 'decent' finite domain? What about smooth curves compared to fractal shapes? And why did it take humanity and the best of its thinkers 2000 years or more to develop Analysis out of greek mathematics? Why are there tasks in Analysis no computer may solve exactly, and what prevents computers from operating on typical real numbers?

2. *Syntax* is the essence of the Calculus which reduces operations in a continuum and rooted in Analysis to *finite algebraic operations.* This algorithmic part avoids dealing with infinities and even the computation of limits is passed on to algebraic manipulations thanks to the Bernoulli-Hôpital rule. The syntax of the Calculus is essential and is at the heart of many tools of modern science and technology. However it is not expected to convey insight and understanding and by its finitistic nature it cannot reach the fundaments of Analysis. Moreover, the available computational power not only allows the construction of reliable CAS software. Numerical procedures usually do not follow from Calculus but rather need the insights of Analysis and algorithmics to be constructed and applied correctly.

   The Calculus is a powerful tool as long as we stay inside the range of *elementary functions.* However, the elementary functions are scarce exceptions in the function spaces dealt with by Analysis. Moreover, the elementary functions are closed with respect to the derivative but *not* with respect to primitive functions. An exclusive use of technology bears the risk, to convey the naive belief that all practically relevant functions are elementary.

No engineer would want to turn back to the slide rule or the trigonometric tables. The tools of the 19th century are definitely outdated. We need creative minds developping and using new tools for the present and the future. Calculus will be among these tools, and Analysis will remain its basis. However it has to be augmented by numerical and statistical methods based on corresponding software and hardware which enable substantial amounts of data to be exploited. Up to the automatised data collection and data processing, data was scarce. The Calculus never showed a big appetite for data. Its central and unrivalled position in the pre-information age may be a consequence of this fact.

Let's open the way for a Calculus enhanced and combined with other algorithms taking advantage of the treasures of big data. This aim cannot be reached without understanding the basic ideas of our intellectual, technological and material tools and heritage.

The use of a CAS in the teaching of mathematics may save some time necessary to dive into the interesting genesis of ideas leading to Analysis. Education that neglects important ideas in favor of drill and practice has to be questionned. We may choose to show big ideas to the next generation with the help of technology like graphing computers, CAS, tablets and more to come. Technology is part of our culture but it will continue to change in contrast to the long lasting and deep ideas in mathematics. We have to take our time to TEACH FUNDAMENTAL IDEAS AND DEMONSTRATE THEIR VALUES ALSO IN APPLICATIONS.

Final comments and pointers to the literature

I have given an account for teaching Analysis and the Calculus according to the genetic approach and my own experience. In line with my motivation, the focus is on a special approach to teaching Analysis at the pre-university level. My message is the combination of the genetic method with the extensive use of technology based on scaffolding and the blackbox–whitebox principles.

I freely admit a somewhat sloppy use of hints to historical facts. The history of the subject is dealt with adequately in [ARN], [EDW], [H&W], [STI].

The *genetic method* was propagated first by Otto Toeplitz since 1926. An authoritative source is [TOE], based on Toeplitz's introductory courses at the university level. David Bressoud, in his preface to [TOE], praises the method as an antidote to the undesired early pseudo-formalisation introduced by New Math. Another text with similar orientation is [SIM].

References

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