Fermi Problems – Solutions and Comments

Remark Solutions to Fermi Problems are subject to additional assumptions not explicitely stated in the problems. Thus the answers are at best approximate and need comments to be useful.

Introductory Examples

E1 A Google search may seem to answer this question quite precisely. The OECD counted 8423 dentists in Belgium in 2008. However, even the term 'dentist working in Belgium' needs further clarification [e.g. working full time, registered in professional society or working in active practice, including accademic teachers, ...].

The number D of dentists in Belgium is limited by various conditions such as

- The total population of Belgium $P \approx 11$ Millions
- socio-economic limitations: Who feeds the dentists? A rule of thumb: in Western societies about 1 in 1500 is a dentist. $D \approx P/1500 \approx 7330$
- preparation of dentists at the medical schools of the country, migration of dentists, ...
  * Student intake: 230 per year
  * Number of graduates: 175 per year, 80% females
  * What is the average lifetime spent as an active dentist and what is the mean working time per year? If a dentist works a total of n years on average, then the replacement rate is $1/n$.
    $D \approx 175n$, $30 \leq n \leq 40$ implies $5250 \leq D \leq 7000$
    [Is there an immigration of dentists? Is the present rate of 175/yr insufficient on longer terms?]
- Political decisions:
  * Minister of Social affairs decides treatment tariffs and oversees relations with sick funds.
  * Minister of Health decides registration, and how many dentists are required.
  * Ministers (2) of Education control the basic education of dental students in each region.
- check-up schedule and general oral health, i.e. how many hours of treatment per person and year are needed on average? [assume $t \approx 1$h/yr]
- how many hours is a dentist working per year and how much time is spent in an average treatment? [assume: 200 days and 7 hours] $\implies D \approx P/1400 \approx 7900$

E2 We simplify the Kepler problem by assuming circular orbits, $M \gg m$ and the centre of mass in the origin of a polar coordinate system. Tehn m orbits around M at a distance r with angular velocity ω.

We compute the total energy of this system whose free parameter we choose to be r. We abreviate $C := \gamma \cdot M \cdot m$, where γ is the universal constant from Newton's law of gravitational attraction.

- potential energy $-C/r$

- kinetic energy $\frac{1}{2}m\omega^2 \cdot r^2$
- condition for circular orbit $C/r^2 = r \cdot \omega^2 m$
- Energy of the system $E(r) = -C/r + \frac{1}{2}m \cdot \omega^2 \cdot r^2 = -\frac{1}{2}C/r$
- Expansion of radius from r to $r + \Delta r$ $\Delta E \approx \dfrac{\mathrm{d}E}{\mathrm{d}r} \cdot \Delta r$ (linearisation)
- Result: $\Delta E = \frac{1}{2}C/r^2 \cdot \Delta r \approx 3.96 \cdot 10^{18}$ J/yr. This energy is equivalent (by Einstein's famous formula) to about 44 kg of mass.

  The increase of the moon's orbital radius by a steady process would require a constant power output of ≈ 126 GW. This is about the output of 130 (big nuclear) power stations.

Solutions to the Questions, Comments

1. (a) $E = 10^9 \cdot 24 \cdot 3600$ J$\approx 8.64 \cdot 10^{13}$ J$\approx 10^{-3}$kg (using $E = m \cdot c^2$ and $c \approx 3 \cdot 10^8$ m/s)

   (b) Any thermic power plant has an efficiency which is majorized by a Carnot machine. The Carnot efficiency is roughly 1/3. Thus the input is at least 3 times the output. The energy consumption of the powerplant is about $2.6 \cdot 10^{14}$ J/d or 3 g/d of energy expressed by its mass equivalent. [NB the Hiroshima bomb had a conversion of less than 1 g of mass into energy and its destructive power was due to the short time during which this energy was released by electromagnetic radiation, particle output, shock waves]

2. We idealise the light rays incident from the sun by assuming that they are parallel close to the earth. This assumption taken literally in a global sense is wrong of course, since it implies that the power of the solar radiation is independent of distance from the source. Furthermore, we neglect the bending of light in the atmosphere.

   The light illuminating a hemisphere is equivalent to the light passing through a (central) cross section S of the earth, perpendicular to the light rays. The power of the solar radiation close to the earth is described by the *solar constant,* $s := 1366$ W/m^2. Neglecting the albedo (back scatter of light from the earth and its atmosphere), the earth collects an amount of $P := s \cdot S$ W of solar power. Using $S = \pi \cdot R^2$, with R the radius of the earth or $R \approx 6.4 \cdot 10^6$ m, we get the *optimistic estimate* $P \approx 1.76 \cdot 10^{17}$ W. This figure may not reach our imagination. So let us compute the answer in mass equivalent: $P \approx 1.96$ kg/s. The power of the sun is roughly equivalent to the explosion of 2000 Hiroshima bombs per second, evenly scattered over one hemisphere of the earth.

   The *albedo* α is variable and by definition $0 \leq \alpha \leq 1$. A usual estimate is $\alpha \approx 1/3$ for the immediate backscatter from clouds, snow or ice covered surfaces, meaning that about 2/3 of P are effective in the atmosphere or on the earth surface, including the oceans. Moreover, the total of the earth surface is four times as big as S. A more realistic estimate for the *average power* available from sun light on a horizontal surface of 1 m^2 on the earth is about $\frac{2}{3}\, s/4 = s/6 \approx 230$W. This amount is further reduced by a factor of at least 10 because solar cells or solar collectors only have a limited efficiency and they need energy for construction and replacement of their own. Like so many of the goods available on earth, solar energy is not evenly distributed over the globe and the typical energy consumption is far from the regions that might be productive in solar energy. Hence further losses are to be considered for the transport of energy and the storage of energy produced from solar radiation.

3. If m is the total mass of the earth's atmosphere and w the mean value of the square of windspeed taken over the whole atmosphere, then $E = \frac{1}{2} \cdot m \cdot w$. We need to estimate m and w. The mass m gets accelerated in the earth's gravitational field by an acceleration of at most g and the resulting force $F = g \cdot m = A \cdot p$, where A is the surface area of the earth and p the standard pressure at sea level. We estimate using paper and pencil $m \approx A \cdot p/g \approx 10^5/10 \cdot 4 \cdot \pi \cdot 6400000^2 \approx 12 \cdot 64^2 \cdot 10^{14} = 3 \cdot 2^{14} \cdot 10^{14} \approx 3 \cdot 2^4 \cdot 10^{17} \approx 5 \cdot 10^{18}$ kg.

   Alternatively, using a calculator and the data $p = 101300$ Pa, $r = 6.380 \cdot 10^6$ m, we get $m \approx 5.182 \cdot 10^{18}$ kg.

   The main difficulty consists in estimating the mean value of v^2 weighted by air density. The top speed of the wind is reached in jet streams at high altitude and hence under minor air density. Winds are mostly rather moderate close to the earth surface. A reasonable estimate for a mean wind might be the velocity by which weather systems are moving on a greater scale. Few of us have a direct intuition for windspeed at 5000 m asl. Now 1 m/s is certainly to small, while 100 m/s is possibly rather high estimate and the geometric mean of 10 m/s might serve as a kind of compromise in absence of better knowledge. Thus we get an estimate for w, the weighted mean of v^2 to be about 100 $\mathrm{m}^2\mathrm{s}^{-2}$ to which we add the correction of 0.5 because it relates to air at an altitude of about 5000 m asl.

   A naive computation now shows $\quad E_{\text{kin}} \approx 0.5 \cdot 0.5 \cdot 10^2 \cdot 5 \cdot 10^{18}\mathrm{J} \quad$ from which we conclude the order of magnitude $\quad 10^{19}\mathrm{J} < E_{\text{kin}} < 3 \cdot 10^{20}$ J.

   Remark This example shows that the essential point consits in a physical insight leading to a simple equation. The estimate using paper and pencil or mental arithmetic – as Fermi would have done – poses an additional difficulty but it doesn't really matter. The main point of the example is an insight into the nature of the problem leading to a simple and simplified relationship fitting our ignorance of precise data.

4. The question reduces to an estimate of the potential energy stored in the form of water inside the clouds of a thunderstorm. Part of this water ends up as precipitation on the ground. There are two bounds: More than the solar energy $E_s$taken up by forming the cloud in a complicated themodynamical process cannot be released. The energy released is at least as big as the potential energy $E - p$ of the precipitation that hit the ground. We estimate the surface of the earth covered by the cloud to be 10×10 km^2 and any topography is neglected.

   (a) We assume that the solar energy is collected during 10 hours with a reduced solar constant of $s/6 \approx 0.2$ kWm^{-2}.
   Then $\quad E_s < 10^8 \cdot 10 \cdot 3600 \cdot 200\mathrm{J} \approx 7.2 \cdot 10^{14} \approx 10^{15}$ J.

   (b) The assumption of the total mass m of water and the mean hight h of the water mass in the cloud determine this estimate: $E_p > m \cdot g \cdot h$ Experience shows that precipitation in thunderstorms can reach 10 liter per meter square to more than 20 liter per meter square. High values are reached often in conjunction with hail stones which indicate that the water has fallen high, possibly 4000 m or more. We assume a height of 2000 m for the value of 10 kg/m^2 and 4000m in case of a mass of 20 kg/m^2. The two cases differ by a factor of 4. Letting $g \approx 10$ m/s^2, we get in the case of water falling from about 2000 m, $E_p > 10^8 \cdot 10 \cdot 10 \cdot 2000 \approx 2 \cdot 10^{13}$ J. In a heavier thunderstorm with the twofold mass and hailstones falling from a double height, we estimate $E_p > 8 \cdot 10^{13}$ J.

5. Data, source Google: radius of the earth $6.38 \cdot 10^6$ m, the oceans cover 71% of the earth surface. The mean depth of the oceans is $h = 3700$ m.
   Volume expansion for water $V(t) = (1 + \gamma \cdot t) \cdot V(0)$, with $\gamma \approx 2.06 \cdot 10^{-4}$ per degree Celsius.

   (a)
   - expansion of the waters due to increased temperature
   - accumulation inside the oceans of mass (rock, sand, water) previously deposited on the continents
   - shrinking of the earth radius

   Basic problem for a model: We assume an increase of temperature by $\Delta t = 1°$C and a constant mass of the waters inside the oceans. Hence the volume of the oceans expands by $\Delta V = \gamma \cdot V = A(t) \cdot \Delta h$ Here the problem becomes evident: We need to know the area of the oceans $A(t)$ as a function of the water temperature. A naive assumption is $A(t) = A =$ const. Then the increase of the mean sea level is $\Delta h = \gamma \cdot h \approx 0.76$m. However, if we assume that the oceans expand uniformly in all directions, then the increase of the sea level is reduced to the linear expansion coefficient $\alpha \approx \gamma/3$ while the area of the oceans increase by $\Delta A \approx 2 \cdot A \cdot \alpha$. Then $\Delta h \approx 0.25$ m.

   We also note that the value adopted for the coefficient for thermic expansion of water refers to a temperature of about 20° C. This condition is fulfilled only in a minor layer of surface water in the subtropical and tropical oceans.

   Because the density of water has a maximum close to 4° C, the expansion coeffient of water is close to 0 in this range. This means that the deep ocean is hardly affected by thermal expansion of water in case of a minor temperature increase $\Delta t = 1°$ C.

   Remark This example shows the risks in naive models. Lacking data like lacking understanding severely limit the value of modelling. Up to now we neglected a possible melting of the ice shields outside the oceans due to increased mean temperatures

   Satellite geodesy has revealed that the surface of the oceans deviates from an idealized ellipsoid in some places by ±60m due to irregularities in the mass distribution inside the earth.

   (b) The volume of the oceans is

   $$V \approx 0.71 \cdot 4\pi \cdot 3700 \cdot 63800000^2 \approx 1.34 \cdot 10^{18} \text{ m}^3$$

   The heath capacity of water is $\kappa \approx 4.2 \cdot 10^3$ J kg^{-1} K^{-1}. Hence the heating up of all the water inside the ocean by 1 K consumes an amount of energy of $\Delta E = \kappa \cdot V \approx 5.6 \cdot 10^{21}$ J. This corresponds to a mass equivalent of $\approx 6.3 \cdot 10^4$ kg of energy. This seems little compared with the energy incident from the sun on earth. The problem of the assumption is rather the homogeneous distribution of this energy inside the oceans. An inhomogeneous distribution of energy is rather compatible with common experience. Unpredictable reactions of the current streaming patterns inside the oceans and the atmosphere are cannot be excluded if the distribution of energy changes. Now, we have reached the stage of speculations beyond the reach of Fermi problems.

(c) Melting of ice deposited on the continents seems to take place. The accumulation of substantial quantities of rain water underneath the deserts has occurred in geologically recent times.

(d) None of the claims may be ruled out on the basis of simple models. The answer 0.25 m seems more plausible than 0.75 m.

This is a typical case where a small 'signal' is hidden behind some substantial amount of 'noise'. In this situation Fermi's method is not to be expected to yield insight.