Graphs, Random Walks, and Matrices
Solutions and Comments

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1 Random Walks, Markov Models

1. Let $\vec{s} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{r} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $M := \begin{bmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{bmatrix}$.

   Then $M^k \cdot \vec{s}$ gives the probability distribution for the two states $\mathbb{S}, \mathbb{R}$ at day k after November 30th coded as state vectors.

   (a) $p_2 = 0.64583$ $p_4 = 0.58435$ $p_8 = 0.57182$ $p_{16} = 0.57143$ $p_{32} = 0.57143$ $p_{33} = 0.57143$

   (b) $q_2 = 0.47222$ $q_4 = 0.55421$ $p_8 = 0.57091$ $q_{16} = 0.57143$ $q_{32} = 0.57143$ $q_{33} = 0.57143$

   Under iteration, the system tends to an equilibrium fixed point. The information about the initial conditions decays quickly.

2. The stochastic vectors in $\mathbb{R}^2$ point from $(0,0)$ to the convex hull of the extrem points $(1,0)$ and $(0,1)$. The stochastic vectors in $\mathbb{R}^3$ point from $(0,0,0)$ to the convex hull of the extrem points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

3. Stochastic vectors have non negative entries whose sum equals 1. Any convex linear combination preserves both properties and the canonical basis consits of stochastic vectors.

4. (a) $S^\intercal \cdot \vec{e} = \vec{e} = 1 \cdot \vec{e}$, 1 is an eigenvalue of $S^\intercal$

   (b) The eigenvalues of S are the same as those of $S^\intercal$.

   (c) $\vec{e}$ is an eigenvector of S if and only if S is doubly stochastic.

   (d) S viewed as a linear map of $\mathbb{R}^3$ is continuous and maps the simplex with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ to itself. Brouwer's fixed point theorem proves the existence of a fixed point among the stochastic vectors.

   (e) Each of the above results generalises to $\mathbb{R}^n$.

5. (a) cf next page

   (b) $\frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   (c) Let U denote the 6×6-matrix all of whose entries are 1. Then $S := \frac{1}{6} \cdot U$ describes the outcomes of rolling a fair dice. The corresponding graph is a labelled 5-simplex (containing 6 vertices) with directed edges between each pair of ordered vertices.

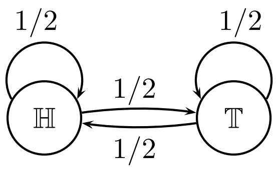

Figure 1: Markov chain model for a fair coin

(d) Analogously in a Laplacian experiment with n outcomes. The matrix is $\frac{1}{n} \cdot U$, where U denotes the $n \times n$-matrix all of whose entries are 1. And the graph corresponds to an $n-1$-simplex with n directed edges each labelled with $1/n$ leaving each vertex and pointing to exactly one of the n vertices. Note that this matrix has rank 1 and only one eigenvalue 1. Equidistribution is an equilibrium configuration. The matrix is symmetric and hence doubly stochastic.

(e) System matrix $B := \begin{bmatrix} q & q \\ p & p \end{bmatrix}$

Equilibrium distribution: $B \cdot \vec{s} := \begin{bmatrix} q \\ p \end{bmatrix}$ for all stochastic $\vec{s}$

2 The Leontjev Toy Model

1. The number 0.4 tells us that from 1 currency unit in sector 2 (agriculture) a fraction of 0.4 is transferred in each time step to sector 1 (industry). Since the model is linear, we may also say that in each time step, 40% of the values present in agriculture are spent in industrial products.

2. The sum in each column of the matrix equals 1. This means that during each time step no money is lost.

3. The money gets distributed repeatedly according to the rules of the linear map given by the Leontjev matrix. The attractive fixed point is due to an eigenvector with eigenvalue 1. The eigenvector with positive entries and summing up to the fixed total of monetary values in the economy tells how the wealth is distributed asymptotically in the future. The relative distribution is given in the stochastic eigenvector

$$\vec{e} \approx \begin{bmatrix} 0.24741 \\ 0.16957 \\ 0.17984 \\ 0.15733 \\ 0.24585 \end{bmatrix}$$

4. Starting value 1 in agriculture spreads into the various sectors of economy and approaches an equilibrium distribution. [cf Table 1]

5. Because of World War II, the US were relatively isolated and mainly producing for domestic consumption and military needs.

6. Leontjev was pioneering in his collection and analysis of data which required an insight into the working of US economy and the interelationship of its many branches. Moreover Leontjev was among the first ones to use computers for treatment and analysis of data.

Table 1: Relative distribution of money in the various sectors after n time steps

steps	0	2	4	8	16
indu	0	0.265	0.24639	0.24741	0.24741
agri	1	0.145	0.17013	0.16957	0.16957
tran	0	0.19	0.17949	0.17984	0.17984
ener	0	0.165	0.15746	0.15733	0.15733
info	0	0.235	0.24654	0.24585	0.24585

He profitted from Harvard's Mark II computer around 1949 in order run matrix models for economy based on his data structuring the US economy into 500 sectors.

Besides modelling closed economies, he developed models for input-output economies.

3 Social Ranking

1. The components of $\vec{v}$ are positive and sum up to ≈ 1, and $G \cdot \vec{v} = \vec{v}$.
2. Think of grooming as giving 'social capital'. Then in the long run, this social capital accumulates in the individuals most worshiped.
3. Using the relative size of the components of $\vec{v}$, we find the ranking of the groups individuals as
$$C > M > B > F > S > E > A > D > G$$
In some cases the hierarchy seems to be questionable. In a group of macaques this is a source of unrest and disputes. We might therefore allow cliques of similar ranking in order to indicate possible alliances or causes for trouble.
$$C > M > B > F > \{S, E\} > \{A, D, G\}$$

Figure 2: Unclear hierarchy

4. Access to a website might express a form of interest and corresponds to grooming.

5. The group of macaques has a well defined collection of individuals. The Internet is dynamic. At any moment some inconsitencies might pop up. The number of websites is constantly changing. There might be references pointing to non-existent sites.

   Any group of animals that may be observed in ecology is small compared to the size of the Internet. The Google matrix is huge and operations on it are time consuming and memory intensive. Numerical problems like round off may prevent a clear linear ranking of the huge number of websites on a scale from 0 to 1. After some time, Google knows approximate answers to the ranking. This opens a possibility for corrections of approximate solutions instead of full computations of eigenvectors.