INTRODUCING MATHEMATICAL CONCEPTS USING COMPUTER ALGEBRA

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## *VISUALISING & EXPLORING with CAS before DEFINING and/or APPLYING*
L.A.A. Blomme – European School Brussels 3
TI-symposium, Brussels – 7/10/2017

In general, there are two ways to use the computational power of the TInspire CAS CX or any CAS based system. The first one is to simply perform the calculations by means of CAS. Avoiding long-lasting calculations saves time and therefore makes it easier to focus on the real problem solving part. More math with less calculations.

But computer algebra can also be put to work didactically. For this purpose, the TInspire CAS software is a great tool. The extensive integration of the various applications in the calculator creates a whole range of new didactic possibilities. By visualising and exploring a problem or a mathematical concept before introducing a rigorous mathematical definition leads to more insight and better understanding.
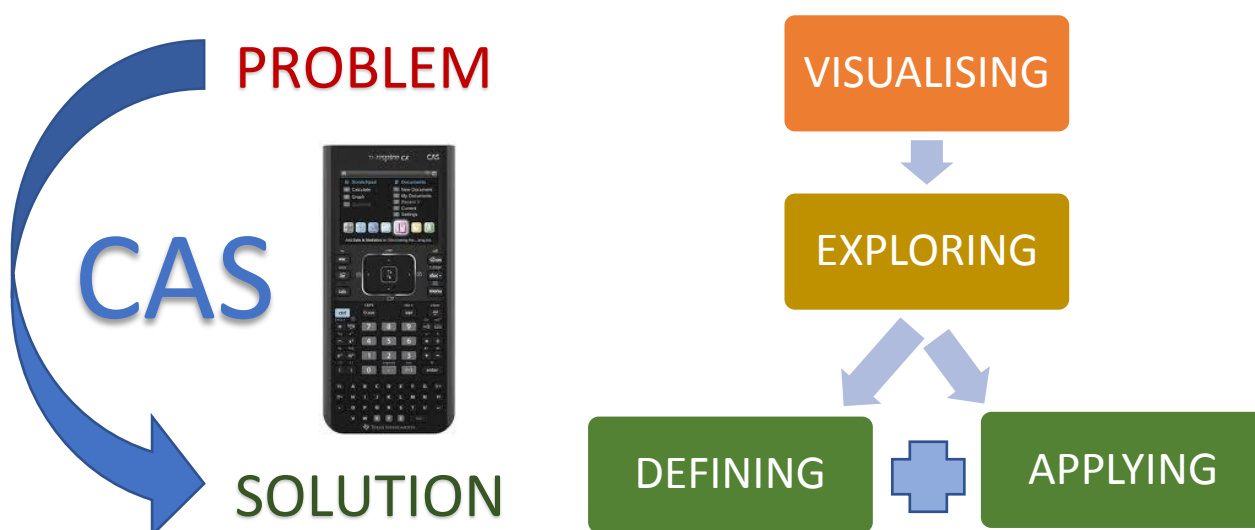
Throughout the following series of examples a notes page is used to define functions and variables, and to perform step by step calculations. Whenever a variable or function is changed, the software automatically adjusts all calculations based on it, on every linked page.

These functions and variables can be displayed using graphs pages – eventually 3D graphing pages. Using sliders, variables on all linked pages can be modified simultaneously on all pages. With sliders, the graphical representation becomes dynamic providing the means for a full exploration of the problem.

Also, the list and spreadsheet application plays an indispensable role in these examples. It is used to perform supporting calculations as well as a buffer to store captured dynamic data while exploring the problem.

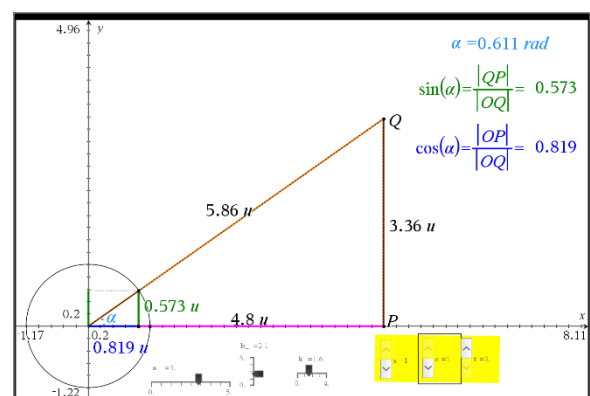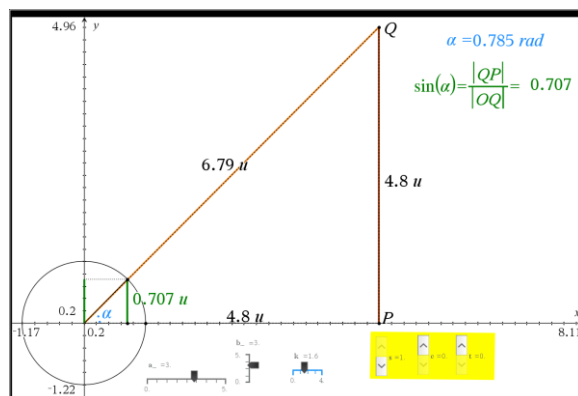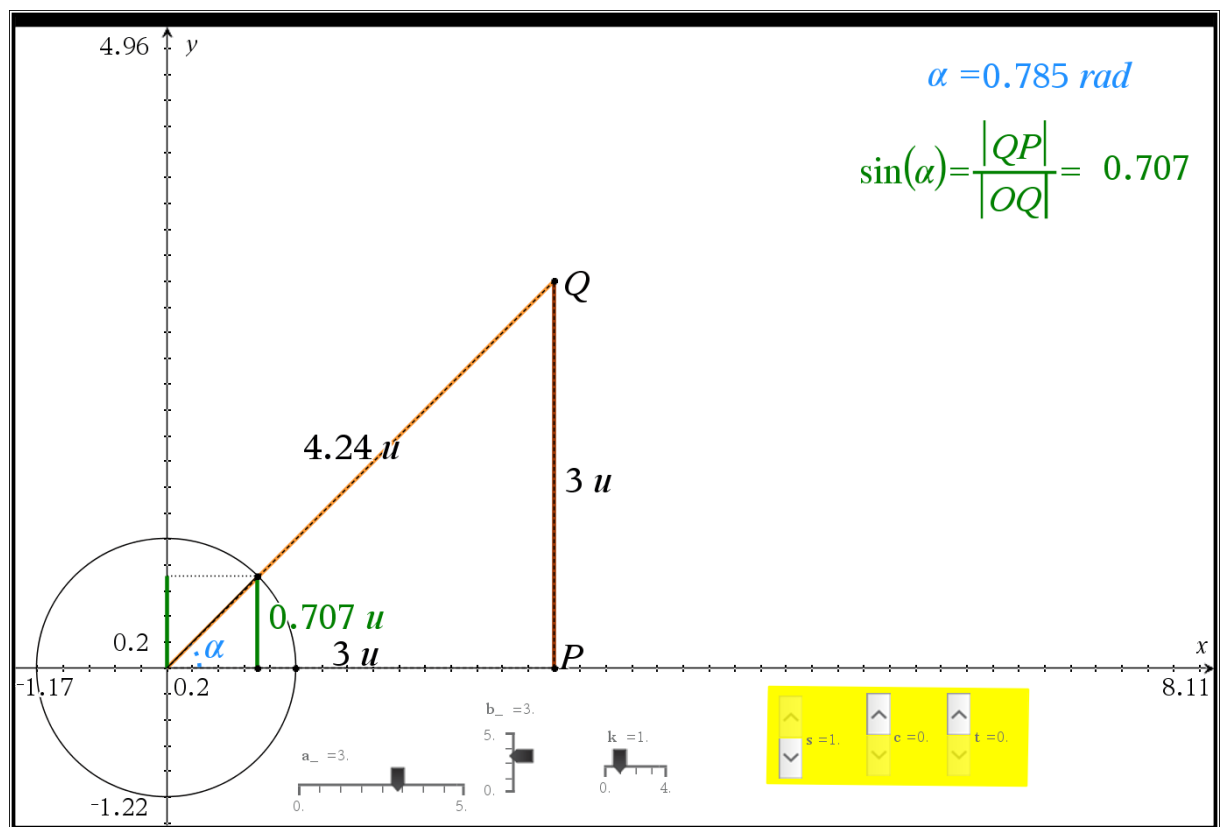# 1. TRIGONOMETRY

1.1 RIGHT-ANGLED TRIANGLE AND UNIT CIRCLE

TNS-file: 1.1 unit circle - triangle

Using the sliders s, c and t, sin, cos and tan can be selected respectively. Of course, you can select more than one function at the same time.

Using the other two sliders a_ and b_ the length of the legs of the right angled triangle a = OP and b = QP can be changed and thus the shape of the right-angled triangle.
Dragging slider k creates a new but similar triangle.
The three trigonometric ratios are measured, calculated and displayed using the right-angled triangle. Applying similar triangles these can also be found on the unit circle.
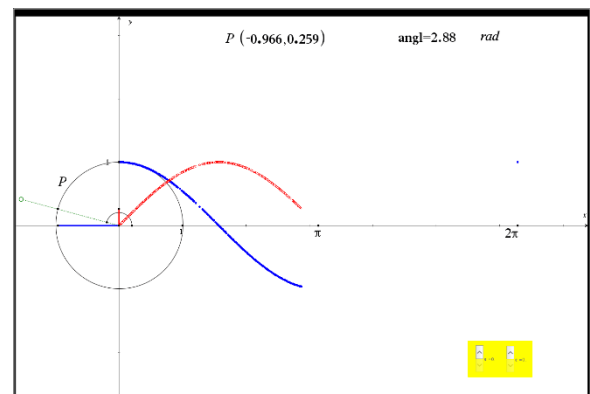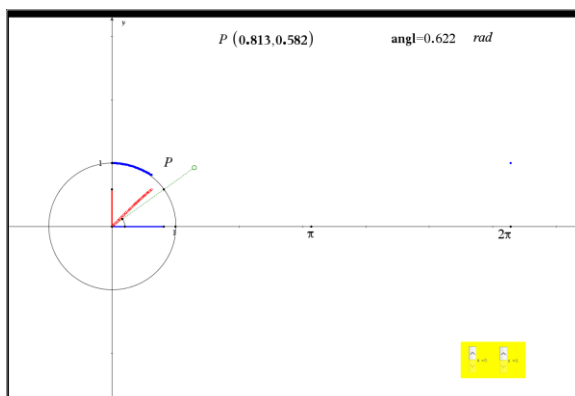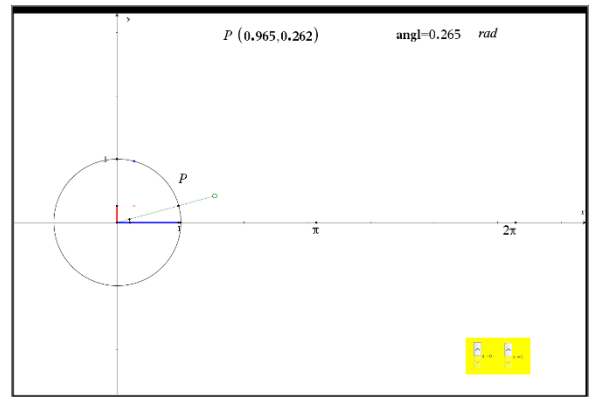
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1.2 TRIGONOMETRIC FUNCTIONS

TNS-file: <mark>1.2 trig functions</mark>

For every position of the point P on the unit circle, it's sin() and cos() – as a projection on the X and Y-axis – are measured and stored into a spreadsheet table.
These values are displayed in real time as a scatter plot.
To change the position of the point P on the unit circle, drag the green handle. As P changes, the scatter plot is adapted to display these changes.

The sliders s and c behave like a switch to hide or display the trigonometric functions sin(x) and/or cos(x).

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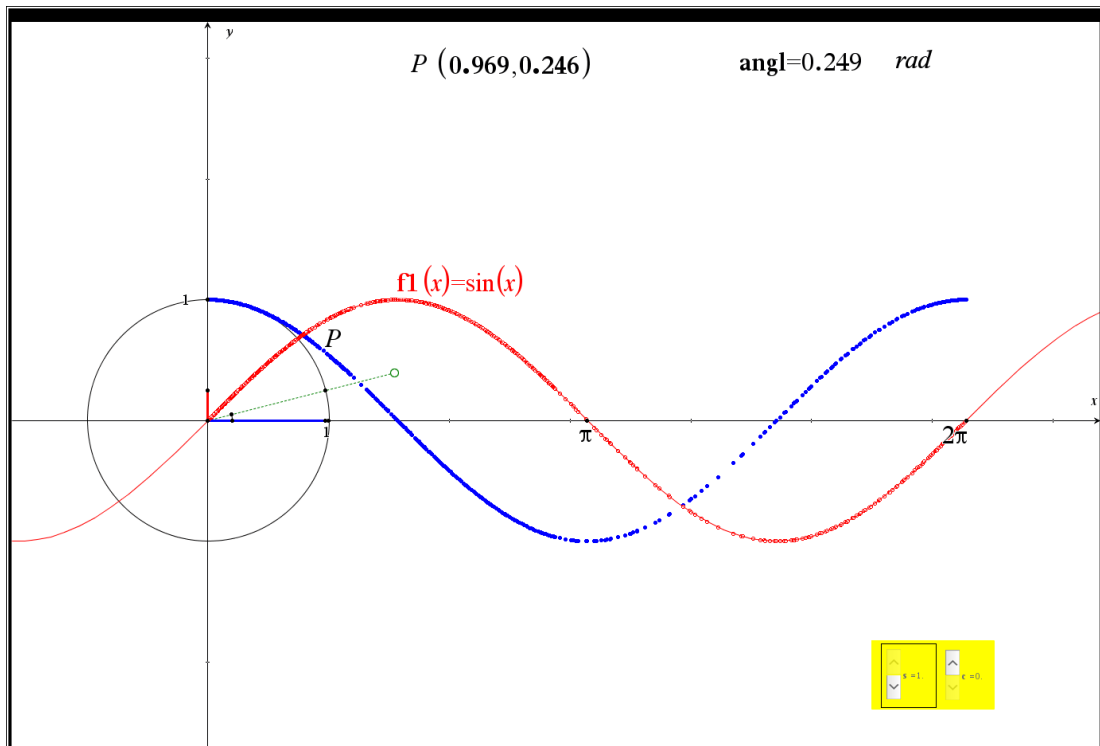
To start the process all over, all data need to be deleted from the supporting spreadsheet. To do so, click every CAPTURE() command and press ENTER twice.

In the second problem of the same file the trigonometric function tan(x) is introduced using the same technique.

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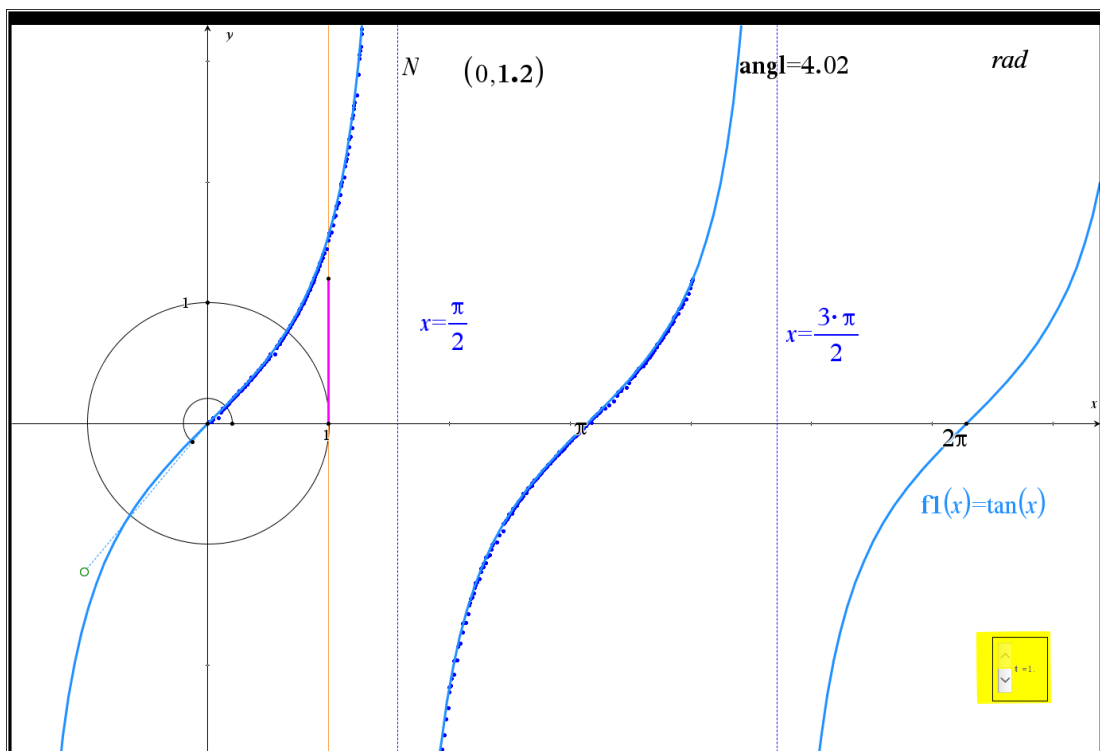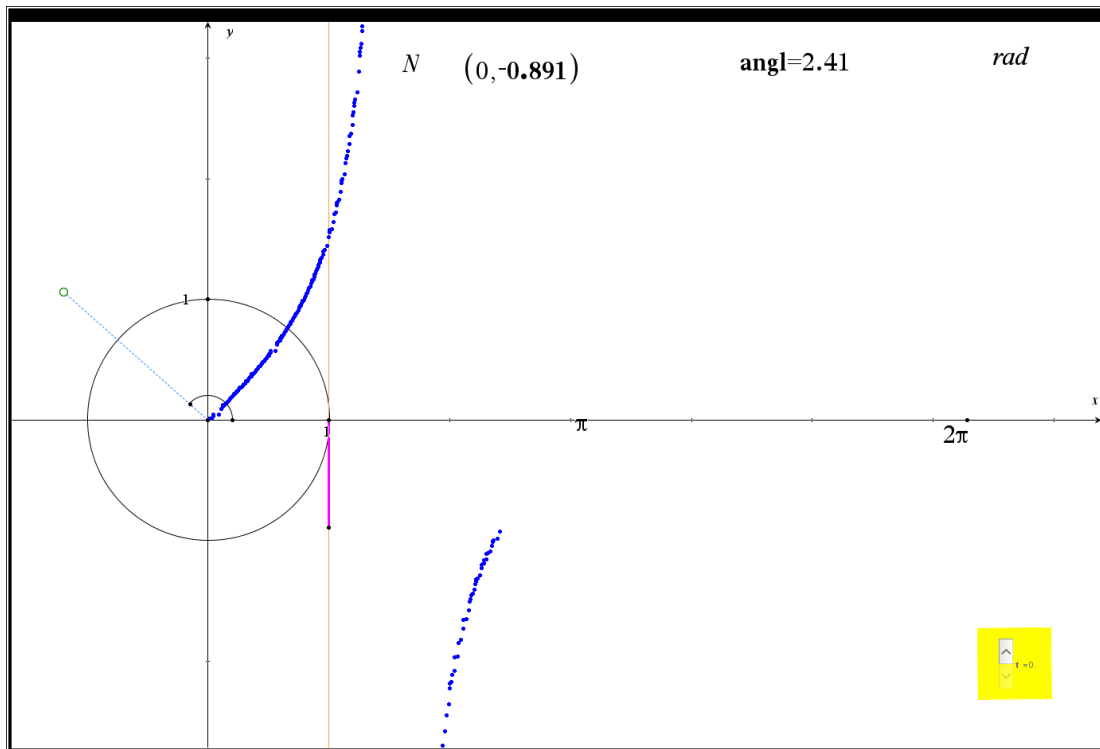
2. COMPLEX NUMBERS

2.1 COMPLEX SQUARE ROOTS

TNS-file: 2.1 COMPL SQUARE ROOTS

A complex number has two complex square roots.
We can show this using a notes page as well as a graphs page. Both pages (of the same problem) are linked and so are all changes and calculations.
Both pages contain 2 sliders p and q to adapt the complex number z = p + i.q .

- z is a positive real number: e.g. z = 4

- z is negative real number: e.g. z = -9

- z is an imaginary: e.g. z = 4i

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- z is a complex number: e.g. z = -4 + 3i

COMPLEX NUMBERS – square roots of p+i·q
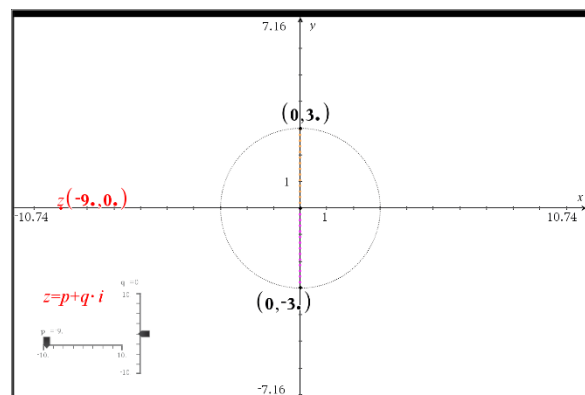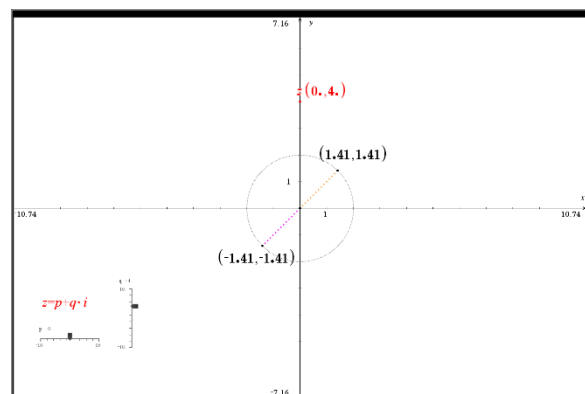
1. Solving in $\mathbb{C}$: $z^2 = p + q \cdot i$
   - drag the point z or use the sliders p and q

   p = -4   en   q = 3

   - calculation:

   $z1 := \sqrt{p + i \cdot q}$ ▸ $0.707107 + 2.12132 \cdot i$
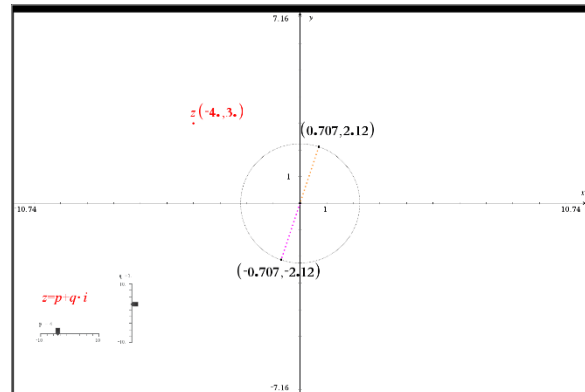 $z2 := -\sqrt{p + i \cdot q}$ ▸ $-0.707107 - 2.12132 \cdot i$
   $x1 := real(z1)$ ▸ 0.707107
   $y1 := imag(z1)$ ▸ 2.12132
   $x2 := real(z2)$ ▸ -0.707107
   $y2 := imag(z2)$ ▸ -2.12132

$z(-4.,3.)$
$(0.707, 2.12)$
$(-0.707, -2.12)$
$z = p + q \cdot i$

2.2 COMPLEX Nth ROOTS

TNS-file: 2.2 COMPL NTH ROOTS

A complex number has n complex roots of degree n.

- E.g. 1: $z^3 = -8$
  All calculations are performed step by step using a notes page.

COMPLEX NUMBERS – solving: $z^n = p + i \cdot q$ / n–th roots

1. Step by step solution in $\mathbb{C}$: $z^3 = -8$
   - given $n := 3$ and $a := -8 \Leftrightarrow z^3 = -8$
   - a solution: $z = r \cdot (\cos(\theta) + i \cdot \sin(\theta))$ and $r > 0$
   - modulus or absolute value: $|a| = 8 \Rightarrow \dfrac{a}{|a|} = -1$

   $\Rightarrow \begin{cases} \cos(\theta) = real\left(\dfrac{a}{|a|}\right) \\ \sin(\theta) = imag\left(\dfrac{a}{|a|}\right) \end{cases}$ or $\{\cos(\theta) = -1, \sin(\theta) = 0\}$

   $\Rightarrow \begin{cases} r^n = |a| \\ n \cdot 0 = angle\left(\dfrac{a}{|a|}\right) + 2 \cdot k \cdot \pi \end{cases}$ or $\{r^3 = 8, 3 \cdot \theta = 2 \cdot k \cdot \pi + \pi\}$

   - $r0 := \sqrt[n]{|a|} = 2$

   and $\theta0 := \dfrac{angle\left(\dfrac{a}{|a|}\right) + 2 \cdot k \cdot \pi}{n} = \dfrac{(2 \cdot k + 1) \cdot \pi}{3} = expand(\theta0) = \dfrac{2 \cdot k \cdot \pi}{3} + \dfrac{\pi}{3}$

   - there are n ▸ 3 different solutions
   $z_0 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=0$ ▸ $z_0 = 1 + \sqrt{3} \cdot i$
   $z_1 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=1$ ▸ $z_1 = -2$
   $z_2 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=2$ ▸ $z_2 = 1 - \sqrt{3} \cdot i$
   - or:
   $z_0 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=0$ ▸ $z_0 = 1. + 1.73205 \cdot i$
   $z_1 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=1$ ▸ $z_1 = -2.$
   $z_2 = r0 \cdot (\cos(\theta0) + i \cdot \sin(\theta0))|k=2$ ▸ $z_2 = 1. - 1.73205 \cdot i$

Using a graphs page all complex nth-roots can be displayed in the Argand plane.

Hint: changing the value of a on the notes page is all you need to do to find a new set of roots. CAS will perform all calculations automatically every time you make a change.

$(1, 1.73)$
$(-2., 0)$
$(1., -1.73205)$
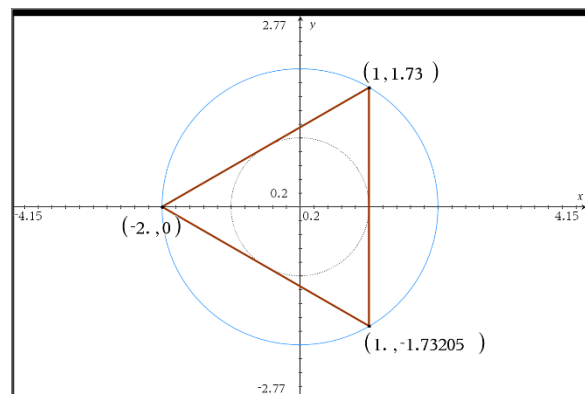
- E.g. 2: $z^4 = 16i$
  Again, all calculations are performed step by step using a notes page.
  To be able to display the complex roots in the complex number plane a list and spreadsheet page is used to calculate the real and imaginary part of every root.

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1. Step by step solution in $\mathbb{C}$: $z^4=16\cdot i$

– given $n:=4$ and $a:=16\cdot i \Leftrightarrow z^4=16\cdot i$

– a solution $z=r\cdot\left(\cos(\theta)+i\cdot\sin(\theta)\right)$ and $r>0$

– modulus: $|a|=16 \Rightarrow \dfrac{a}{|a|}=i$

$$\Rightarrow \begin{cases}\cos(\theta)=\text{real}\left(\dfrac{a}{|a|}\right)\\ \sin(\theta)=\text{imag}\left(\dfrac{a}{|a|}\right)\end{cases} \text{ or } \left\{\cos(\theta)=0,\sin(\theta)=1\right\}$$

$$\Rightarrow \begin{cases}r^n=|a|\\ n\cdot\theta=\text{angle}\left(\dfrac{a}{|a|}\right)+2\cdot k\cdot\pi\end{cases} \text{ or } \left\{r^4=16,4\cdot\theta=2\cdot k\cdot\pi+\dfrac{\pi}{2}\right\}$$

– $r0:=\sqrt[n]{|a|}=2$

and $\theta0:=\dfrac{\text{angle}\left(\dfrac{a}{|a|}\right)+2\cdot k\cdot\pi}{n}=\dfrac{(4\cdot k+1)\cdot\pi}{8}=\text{expand}(\theta0)=\dfrac{k\cdot\pi}{2}+\dfrac{\pi}{8}$

– There are $n\blacktriangleright 4$ different solutions/roots

$z_0=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=0 \blacktriangleright z_0=\sqrt{\sqrt{2}+2}+\sqrt{2-\sqrt{2}}\cdot i$

$z_1=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=1 \blacktriangleright z_1=-\sqrt{2-\sqrt{2}}+\sqrt{\sqrt{2}+2}\cdot i$

$z_2=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=2 \blacktriangleright z_2=-\sqrt{\sqrt{2}+2}-\sqrt{2-\sqrt{2}}\cdot i$

$z_3=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=3 \blacktriangleright z_3=\sqrt{2-\sqrt{2}}-\sqrt{\sqrt{2}+2}\cdot i$

– or:

$z_0=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=0 \blacktriangleright z_0=1.84776+0.765367\cdot i$

$z_1=r0\cdot\left(\cos(\theta0)+i\cdot\sin(\theta0)\right)|k=1 \blacktriangleright z_1=-0.765367+1.84776\cdot i$

2. checking

– argument of a = a : $\text{angle}(a) \blacktriangleright \dfrac{\pi}{2}$

$\Rightarrow \theta0 = \text{angle}(a)/n \blacktriangleright \dfrac{\pi}{8}$

– nth root (only one is calculated): $\sqrt[n]{a} \blacktriangleright \sqrt{\sqrt{2}+2}+\sqrt{2-\sqrt{2}}\cdot i$

– all solutions: $\text{cSolve}\left(z^n=a,z\right)$

$z=-\sqrt{2-\sqrt{2}}+\sqrt{\sqrt{2}+2}\cdot i$ or $z=\sqrt{2-\sqrt{2}}-\sqrt{\sqrt{2}+2}\cdot i$ or

$z=-\sqrt{\sqrt{2}+2}-\sqrt{2-\sqrt{2}}\cdot i$ or $z=\sqrt{\sqrt{2}+2}+\sqrt{2-\sqrt{2}}\cdot i$

– or : $\text{cPolyRoots}\left(z^n-a,z\right)$

$\left\{-1.84776-0.765367\cdot i,-0.765367+1.84776\cdot i,0.765367-1.84776\cdot i,1.847\right.$

(plot) $(-0.759,1.85)$ $(1.84776,0.765367)$ $(-1.84776,-0.765367)$ $(0.758,-1.85)$
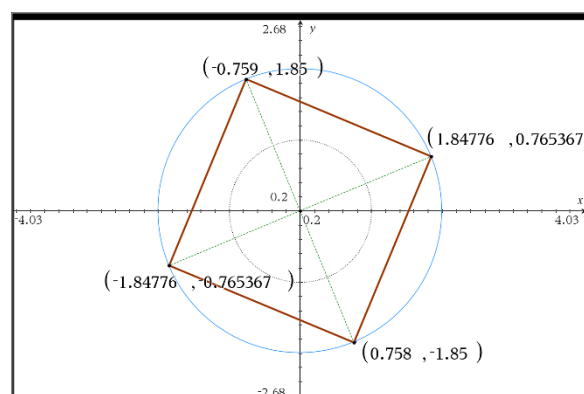
- E.g. 3: $z^n = p + iq$

  Use the 3 sliders n, p and q to adapt the value of n as well as the complex number p+i.q. The complex roots are recalculated and displayed in real time.

1. Step by step solution in $\mathbb{C}$: $z^n=p+q\cdot i$

– see graphs page: $n \blacktriangleright 7$.   and $a:=p+q\cdot i \Leftrightarrow z^7=8.-4.\cdot i$

– drag slider n

– drag point z or drag the sliders p and q

– all solutions: $\text{cSolve}\left(z^n=a,z\right)$

$z=1.36453-0.090512\cdot i$ or $z=0.921533+1.0104\cdot i$ or

$z=0.780002-1.12326\cdot i$ or $z=-0.215393+1.35045\cdot i$ or

$z=-0.391878-1.31017\cdot i$ or $z=-1.19012+0.673594\cdot i$ or

$z=-1.26867-0.510497\cdot i$

– or : $\text{cPolyRoots}\left(z^n-a,z\right)$

– all solutions: $\text{cSolve}\left(z^n=a,z\right)$

$z=1.36453-0.090512\cdot i$ or $z=0.921533+1.0104\cdot i$ or

$z=0.780002-1.12326\cdot i$ or $z=-0.215393+1.35045\cdot i$ or

$z=-0.391878-1.31017\cdot i$ or $z=-1.19012+0.673594\cdot i$ or

$z=-1.26867-0.510497\cdot i$

– or : $\text{cPolyRoots}\left(z^n-a,z\right)$

$\left\{-1.26867-0.510497\cdot i,-1.19012+0.673594\cdot i,-0.391878-1.31017\cdot i,-0.21\right.$

	A wortels	B aa	C bb	D	E	F	G
=	=approx(cpolyroots(z^('n)−'a,z))	=real(a[])	=imag(a[])				
1	-1.2686667725812−0.51049669...	-1.26867	-0.510497	1.36752			
2	-1.190123181039+0.673594141...	-1.19012	0.673594				
3	-0.3918784082838−1.31017310...	-0.391878	-1.31017				
4	-0.21539256008578+1.3504548...	-0.215393	1.35045				
5	0.780002390314−1.123262446...	0.780002	-1.12326				
6	0.92153305181951+1.0103955...	0.921533	1.0104				
7	1.3645254798562−0.09051225...	1.36453	-0.090512				
8							
9							
10							
11							
12							
13							
14							
15							

$A1 = -1.2686667725812−0.51049669312073\cdot i$

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2.3 COMPLEX TRANSFORMATIONS IN THE ARGAND PLANE

TNS-file: 2.3 COMPL TRANSFORMATIONS
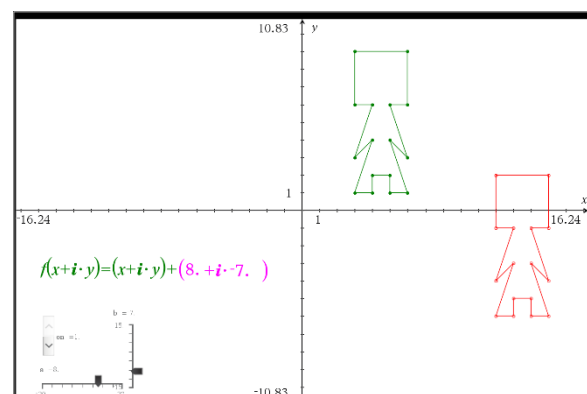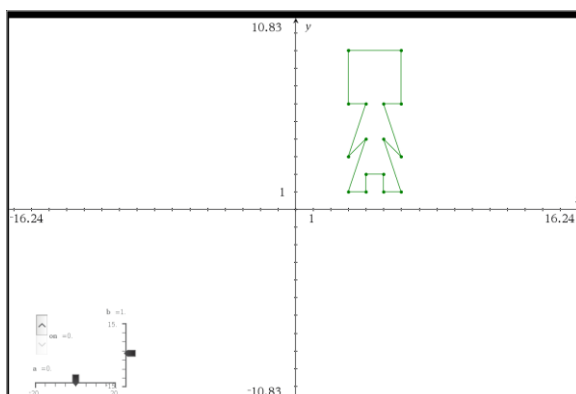
- E.g. 1: $f(z) = z + (a + ib)$ - translation operator

TRANSFORMATIONS IN THE ARGAND PLANE

- translation operator t(a,b)
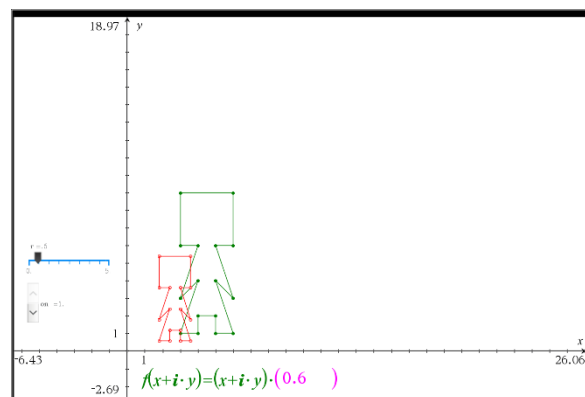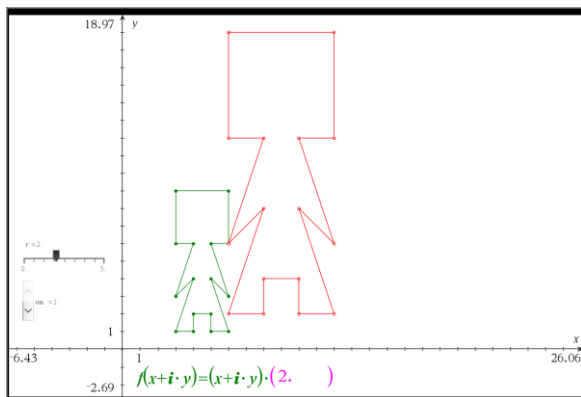- define the complex tranformation: $f(z):=z+a+i \cdot b$

1. See graphs page
   - drag sliders a and b to change the vector of the translation
   - click the switch on to hide or display the shifted image

$f(x+i \cdot y) = (x+i \cdot y) + (8. + i \cdot {}^{-}7.\ )$

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- E.g. 2: $f(z) = r.z$ and $r \in \mathbb{R}_0^+$ - scaling

$f(x + i \cdot y) = (x + i \cdot y) \cdot (2. \quad)$

$f(x + i \cdot y) = (x + i \cdot y) \cdot (0.6 \quad)$

- E.g. 3: $f(z) = \overline{z}$ - reflection in the X-axis
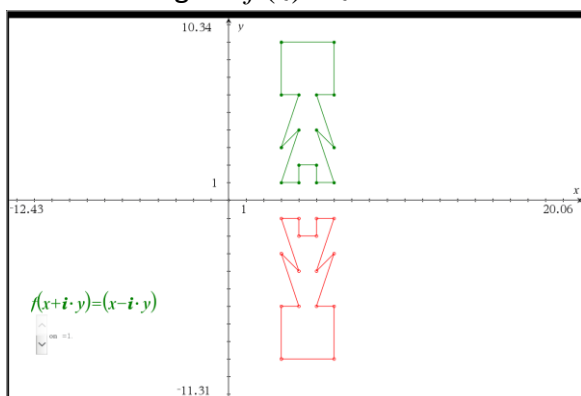
$f(x + i \cdot y) = (x - i \cdot y)$

- E.g. 4: $f(z) = z.(\cos \alpha + i.\sin \alpha)$ - rotation centred at O(0,0)

$f(x + i \cdot y) = (x + i \cdot y) \cdot (-3.67\text{E}-6 + i \cdot 1. \quad)$

$f(x + i \cdot y) = (x + i \cdot y) \cdot (0.284 + i \cdot -0.959)$
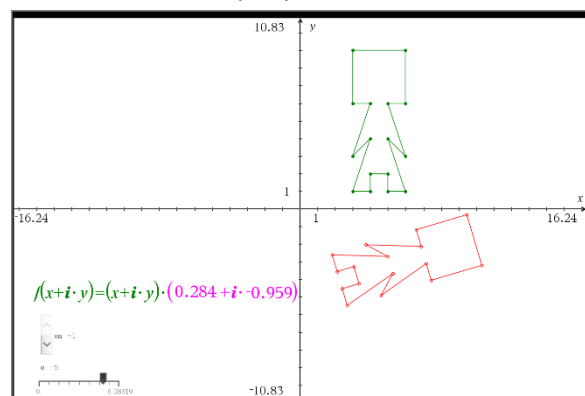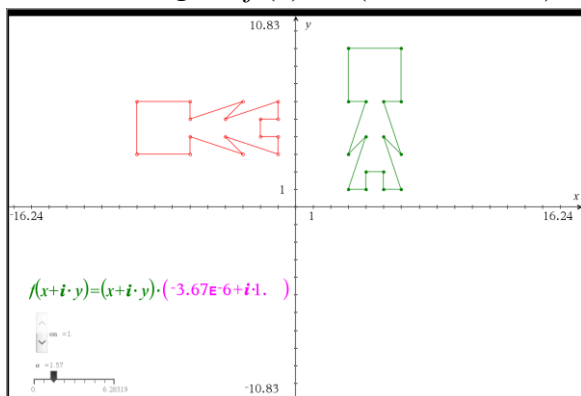
- E.g. 5: $f(z) = f(x + i.y) = -x + i.y$
  - reflection in the Y-axis

text

$f(x + i \cdot y) = (-x + i \cdot y)$

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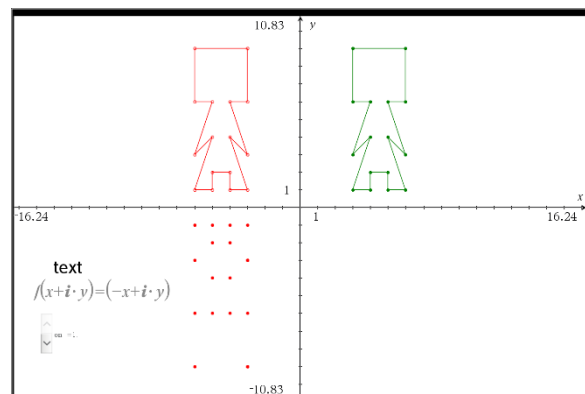
- E.g. 6: $f(z) = (z - a - ib).(\cos\alpha + i.\sin\alpha) + a + ib$ - rotation centred at P(a,b) with scaling r

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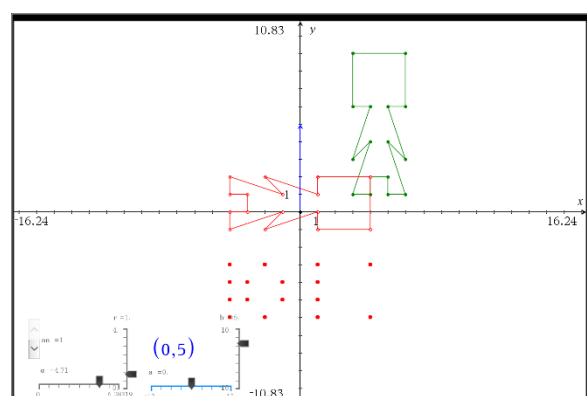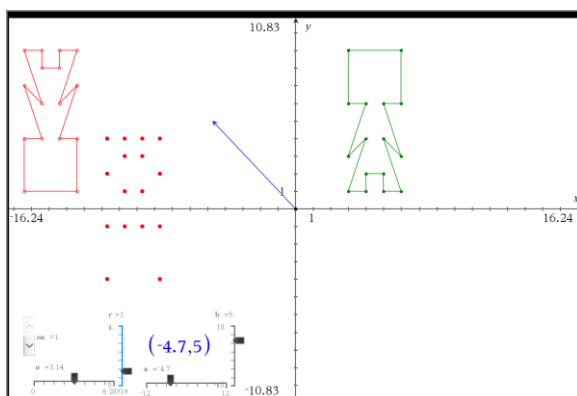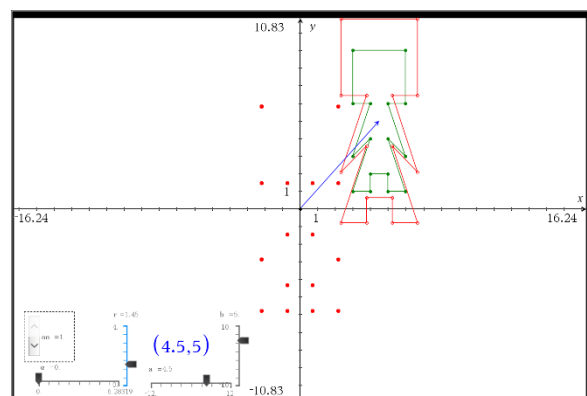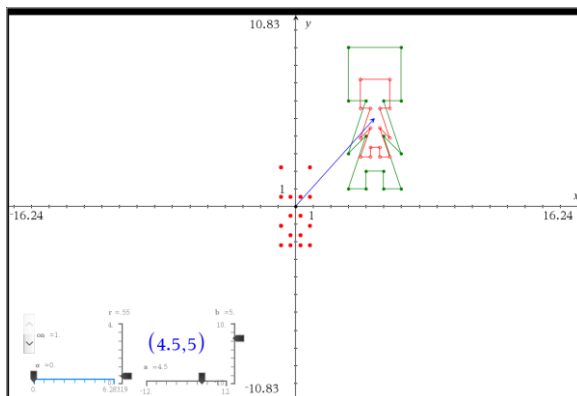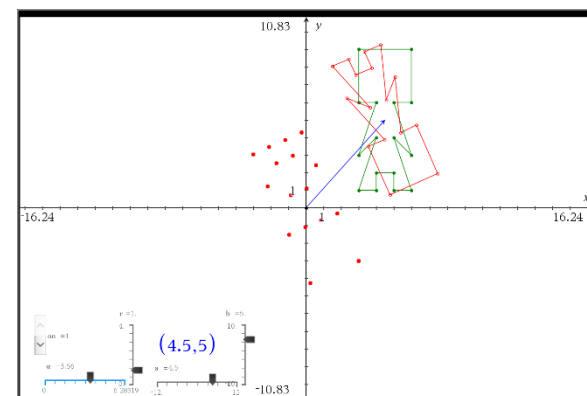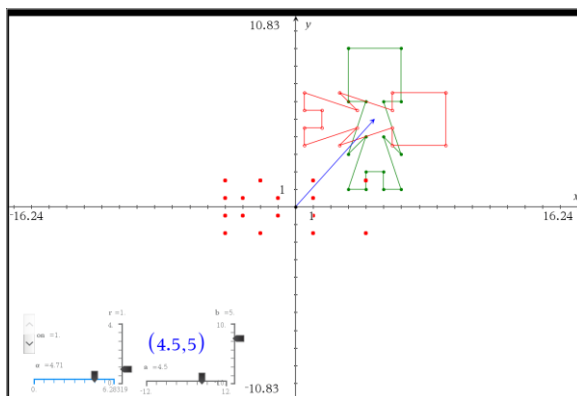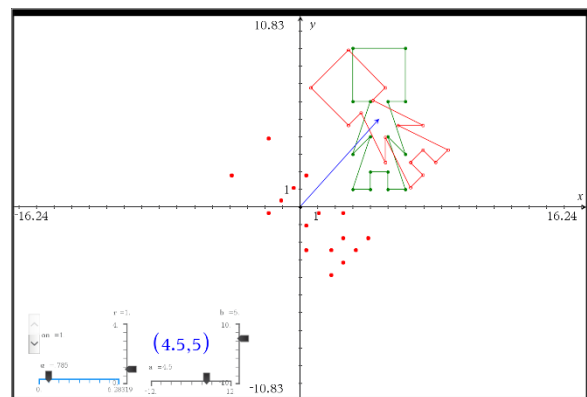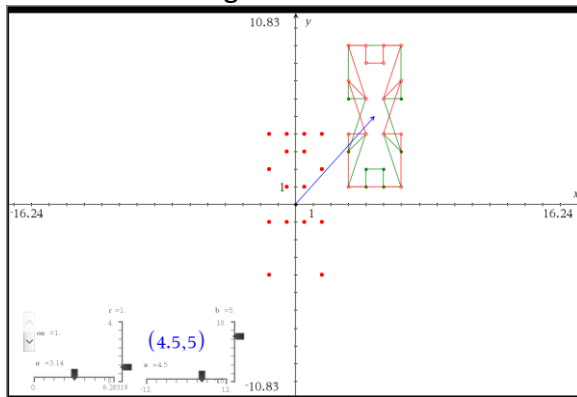
3. REAL FUNCTIONS

3.1 DOMAIN and RANGE

TNS-file: <mark>3.1 FUNCTIONS DOMAIN RANGE</mark>

Once a function is defined on a notes page, the command DOMAIN(f(x),x) – executed in a math box - will display the domain of this function.

P is a point on the graph of the function already defined on the notes page. Dragging this point P on the graph will create a graphical representation of the domain as well as the range of the function using a perpendicular projection of the point P on the X and Y-axis.
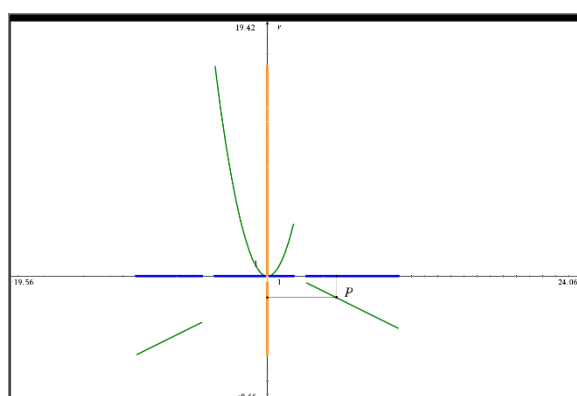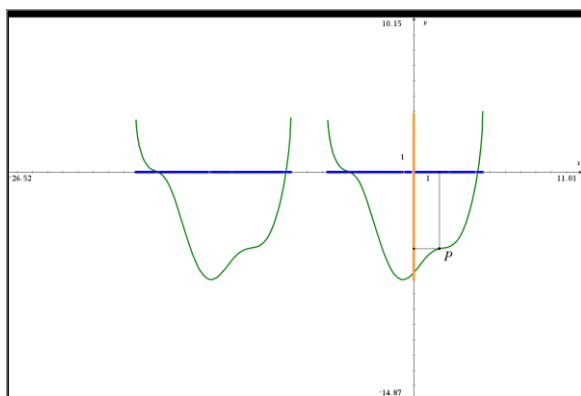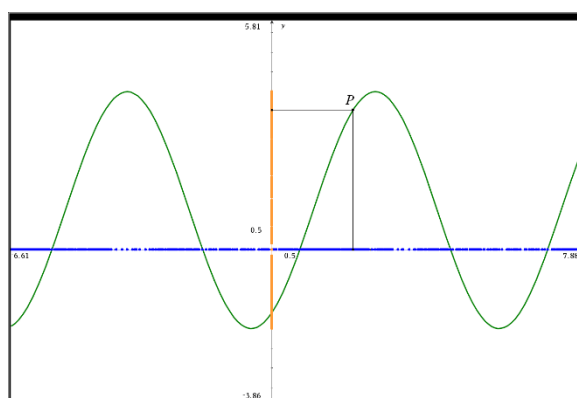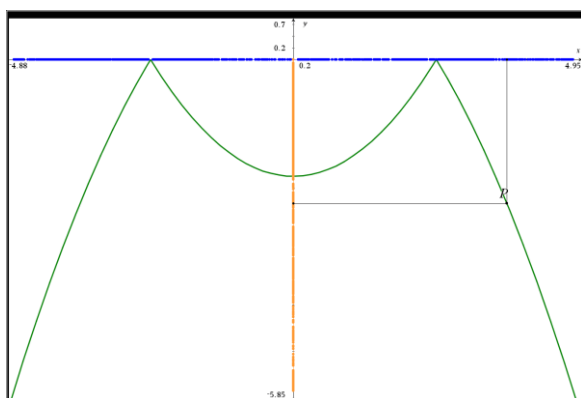
The TNS file contains many more examples.

3.2 PARITY: ODD or EVEN FUNCTIONS

TNS-file: <mark>3.2 FUNCTIONS ODD EVEN</mark>

As usual, the function is defined on a notes page. By simply changing the definition of the function another function can be studied.

This application focusses on finding even and/or odd variations of a given function. How many are there, etc.?

The graph of the function is displayed on a graphs page. The slider on works as a switch with 3 positions: 0 = display nothing; 1 = display first solution; 2 = display second solution.

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- E.g. 1: even variations
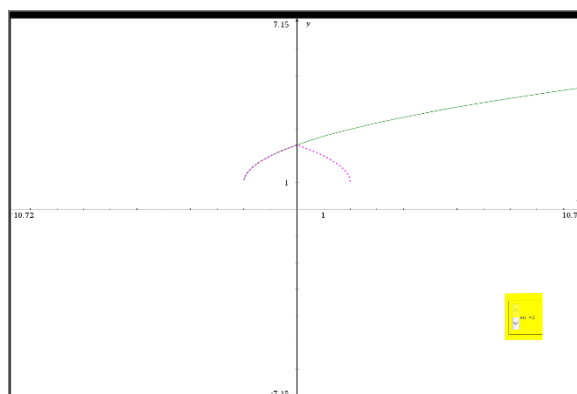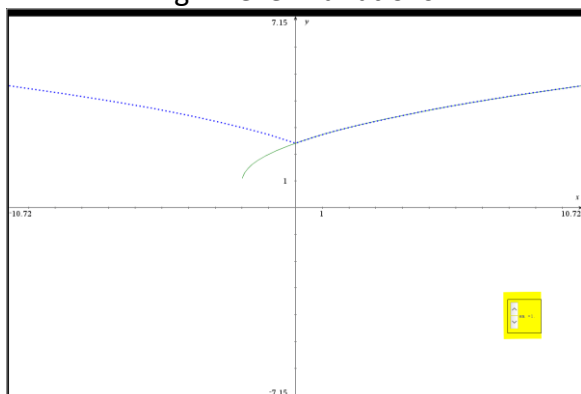
- E.g. 2: odd variations
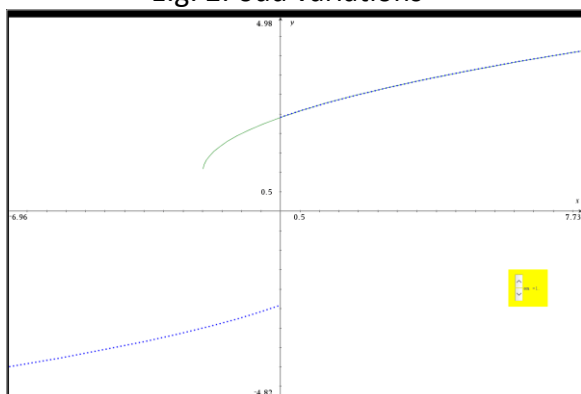
3.3 OPERATIONS with functions

- SUM of two functions

TNS-file: <mark>3.3a SUM OF 2 FUNCTIONS</mark>

Both functions are defined on a notes page and displayed on the same graphs page. Dragging the P on the X-axis creates a scatter plot of the sum of the two functions.
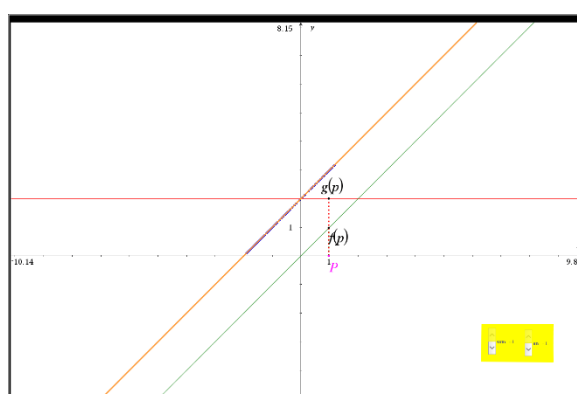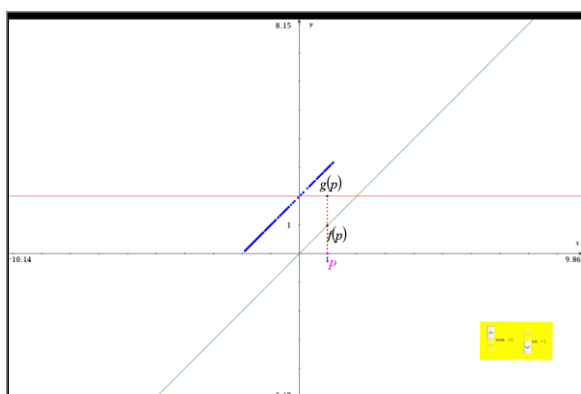

The graphs page contains two sliders which function as a switch. Slider sum switches between displaying and hiding the sum of the two functions. Slider on switches the scatterplot on or off.

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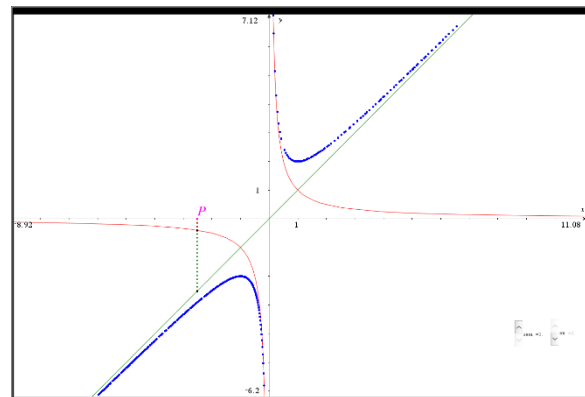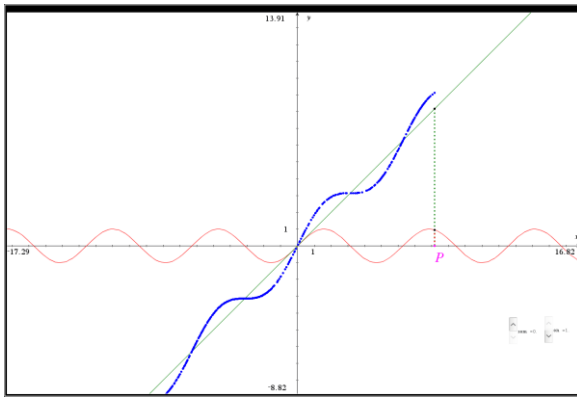
- Product of two functions: scaling, a special case

TNS-file: 3.3b PRODUCT OF 2 FUNCTIONS - spec

One of the functions is a constant function.
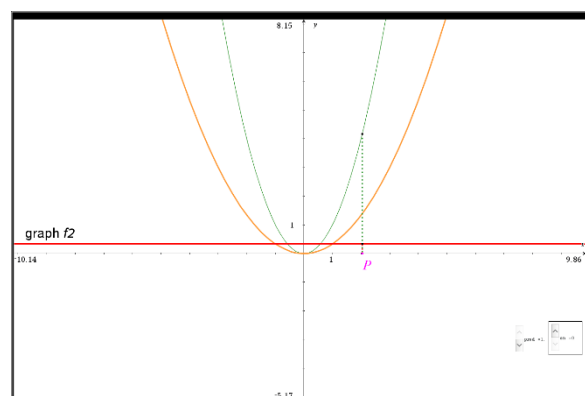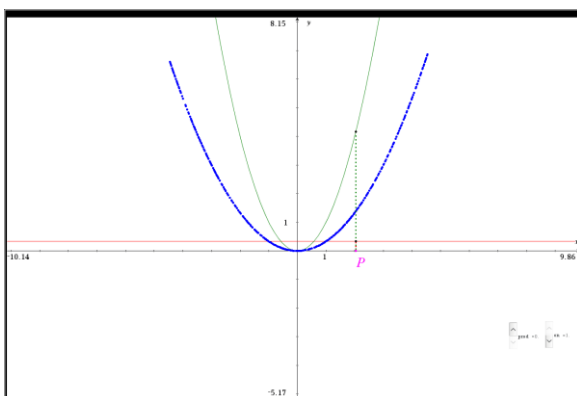
- Product of two functions: special case

TNS-file: 3.3c PRODUCT OF 2 FUNCTIONS
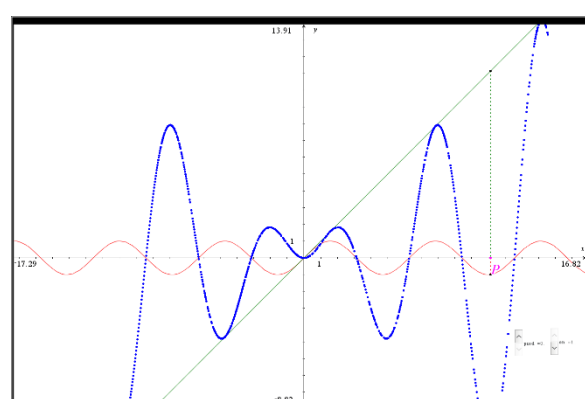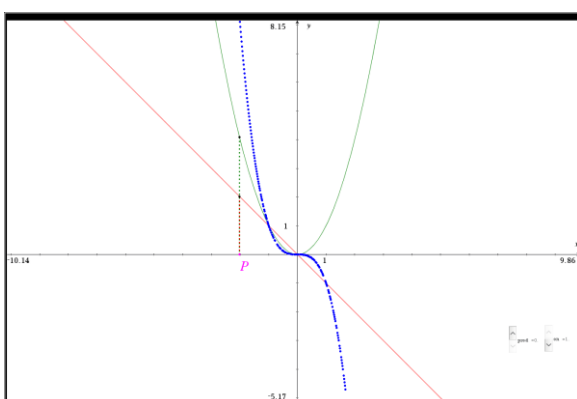
- QUOTIENT OF 2 FUNCTIONS

  TNS-file: 3.3d QUOTIENT OF 2 FUNCTIONS

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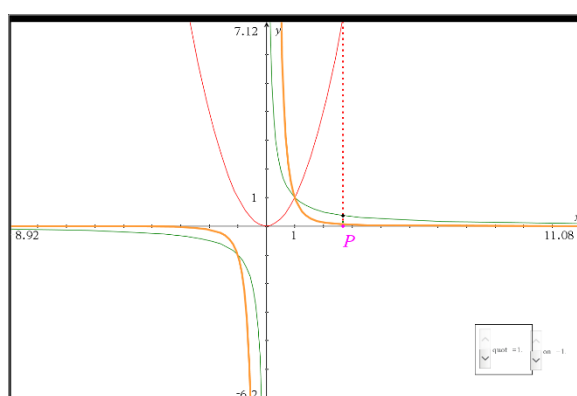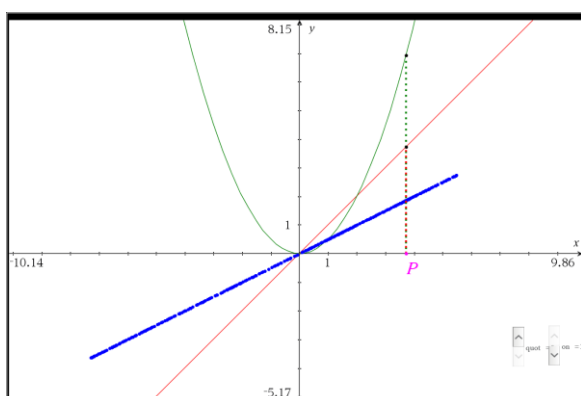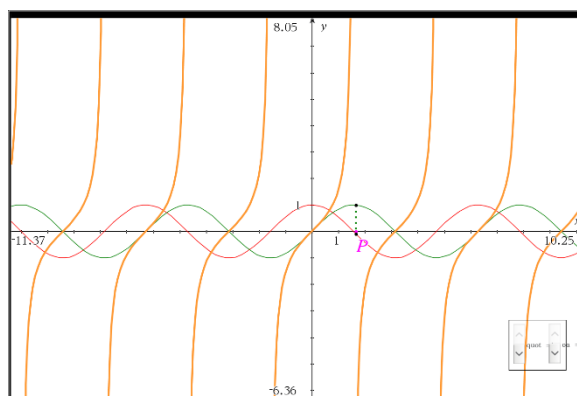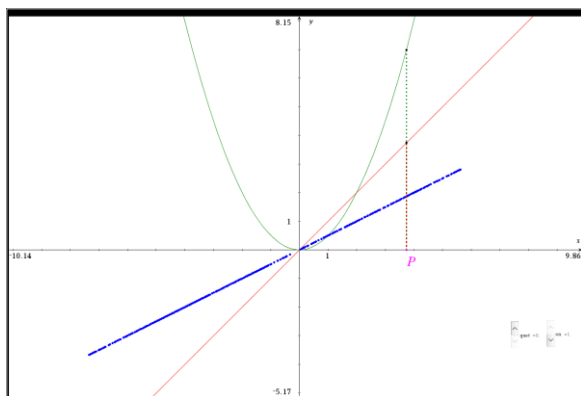
This file contains a lot of other examples.

4. RELATIVE POSITION OF TWO SPHERES – POINTS OF INTERSECTION

TNS-file: 4.1 2 SPHERES

This file contains an overview of all possible relative positions of two spheres. It includes a step by step calculation of the points of intersection as well as a 3D graph of the relative position of the spheres. The definition of the spheres and all calculations are performed on a notes page.

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By placing the centre of both spheres on the X-axis, every sphere becomes a solid of revolution obtained by rotating a half circle (situated in a plane) about the X-axis.

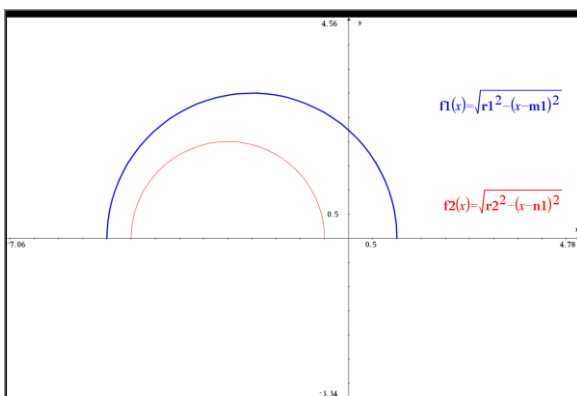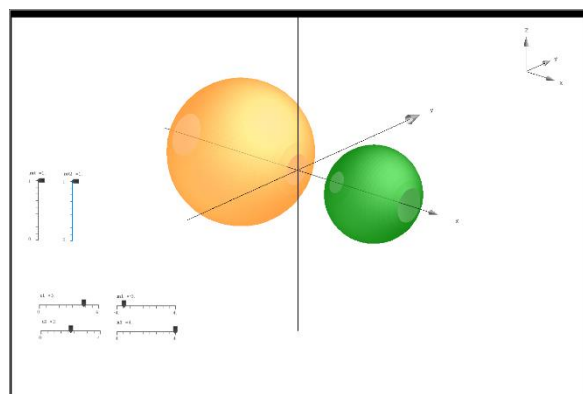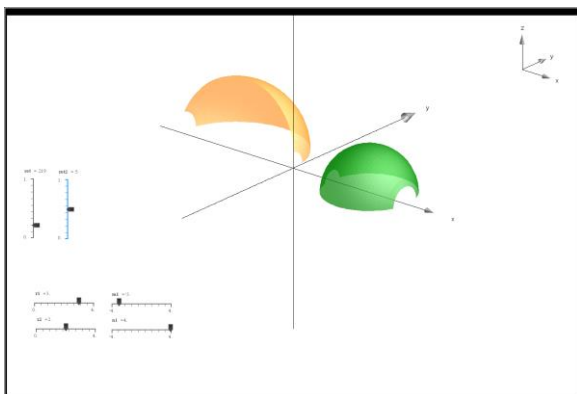The 3D graphing page contains 6 sliders:

- Use the sliders m1 and n2 to change the position of the centre of a sphere on the X-axis.
- Use the sliders r1 and r2 the changes the radius of a sphere.
- The sliders Rot and rot2 are used to control the display of the rotation process.

Using these sliders we can explore the different relative positions of two spheres.
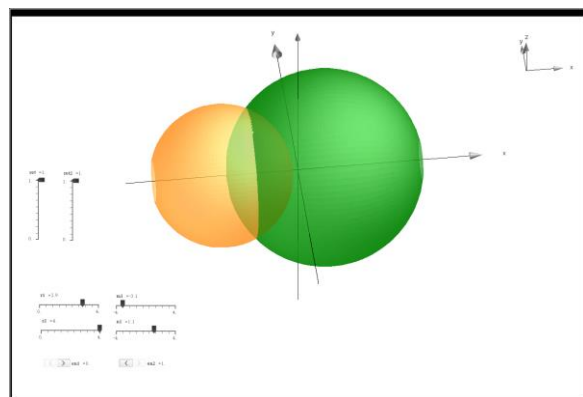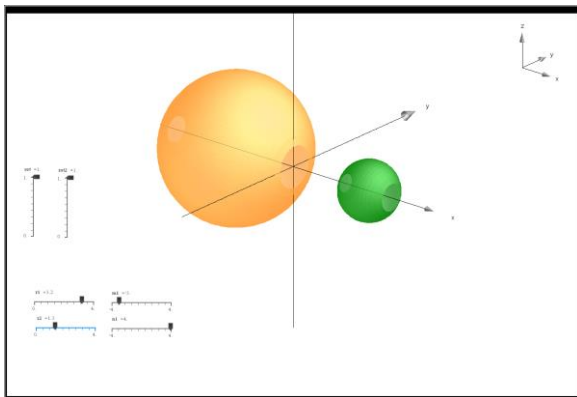
To calculate the points of intersection, a different approach is needed for every case.

We will only consider to spheres with a different radius.

- Case 1: no points of intersection

$$f1(x) = \sqrt{r1^2 - (x - m1)^2}$$

$$f2(x) = \sqrt{r2^2 - (x - n1)^2}$$

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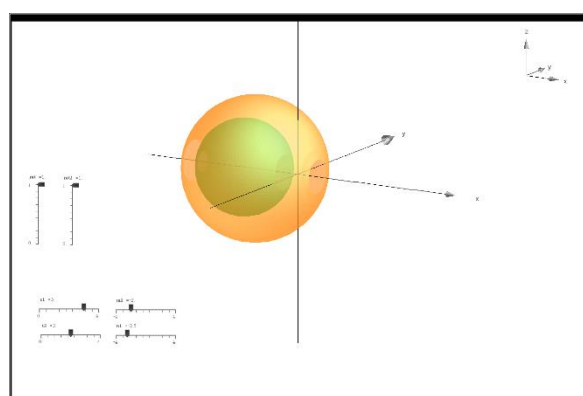
- Case 2: 2 tangent spheres – one point of intersection

2. calculation:
- $d(B_1, B_2) = \mathrm{norm}(\mathbf{m-n}) \cdot 5 = \mathbf{r1+r2} \cdot 5 \Rightarrow$ tangent spheres
- one point of intersection (tangent point)
  - tangent point = point of intersection of the line NM with B1 and B2
  - line NM: $\mathbf{l} := \lambda \cdot (\mathbf{m-n}) + \mathbf{n} \cdot \begin{bmatrix} 3-5 \cdot \lambda \\ 0 \\ 0 \end{bmatrix}$
  - points of intersection with B1:
    $\mathrm{solve}\left((3-5\cdot\lambda-\mathbf{m1})^2+(0-\mathbf{m2})^2+(0-\mathbf{m3})^2=\mathbf{r1}^2,\lambda\right) \cdot \lambda=\frac{2}{5}$ or $\lambda=\frac{8}{5}$
    $\Rightarrow \mathbf{l}|\lambda=\frac{2}{5} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{l}|\lambda=\frac{8}{5} \cdot \begin{bmatrix}-5\\0\\0\end{bmatrix}$
  - points of intersection with B2:
    $\mathrm{solve}\left((3-5\cdot\lambda-\mathbf{n1})^2+(0-\mathbf{n2})^2+(0-\mathbf{n3})^2=\mathbf{r2}^2,\lambda\right) \cdot \lambda=\frac{-2}{5}$ or $\lambda=\frac{2}{5}$

- points of intersection with B1:
  $\mathrm{solve}\left((3-5\cdot\lambda-\mathbf{m1})^2+(0-\mathbf{m2})^2+(0-\mathbf{m3})^2=\mathbf{r1}^2,\lambda\right) \cdot \lambda=\frac{2}{5}$ or $\lambda=\frac{8}{5}$
  $\Rightarrow \mathbf{l}|\lambda=\frac{2}{5} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{l}|\lambda=\frac{8}{5} \cdot \begin{bmatrix}-5\\0\\0\end{bmatrix}$
- points of intersection with B2:
  $\mathrm{solve}\left((3-5\cdot\lambda-\mathbf{n1})^2+(0-\mathbf{n2})^2+(0-\mathbf{n3})^2=\mathbf{r2}^2,\lambda\right) \cdot \lambda=\frac{-2}{5}$ or $\lambda=\frac{2}{5}$
  $\Rightarrow \mathbf{l}|\lambda=\frac{2}{5} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{l}|\lambda=\frac{-2}{5} \cdot \begin{bmatrix}5\\0\\0\end{bmatrix}$
- intersection point of B1 and B2 $= \begin{bmatrix}1\\0\\0\end{bmatrix}$
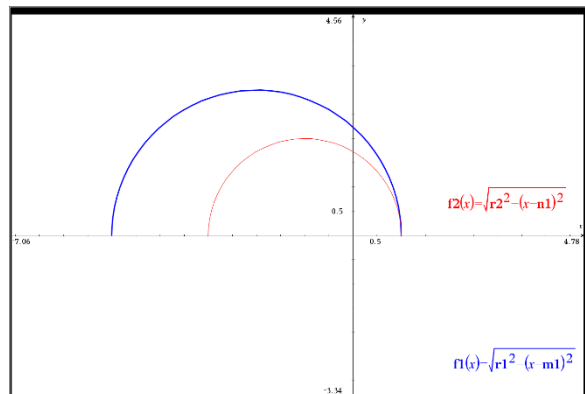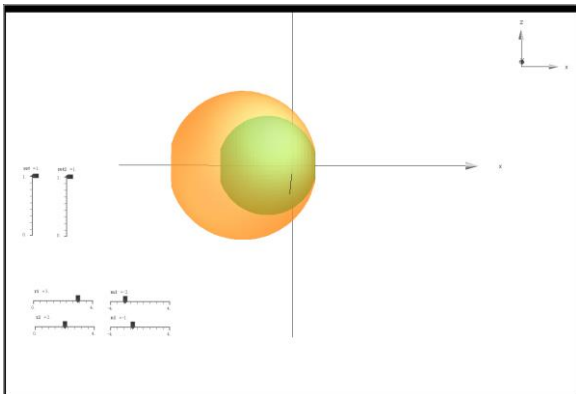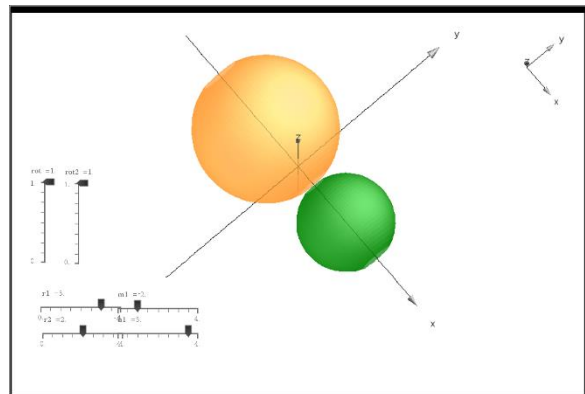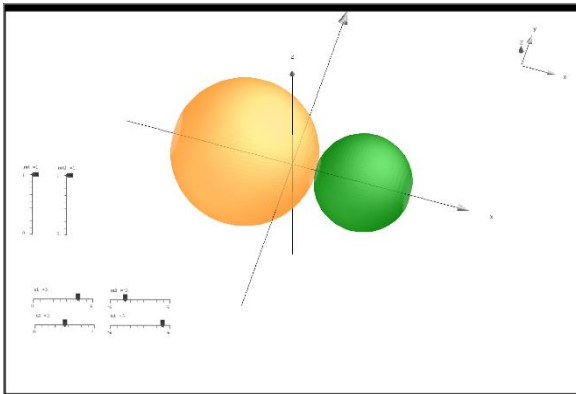
- Case 3: 2 intersecting spheres – circle of intersection

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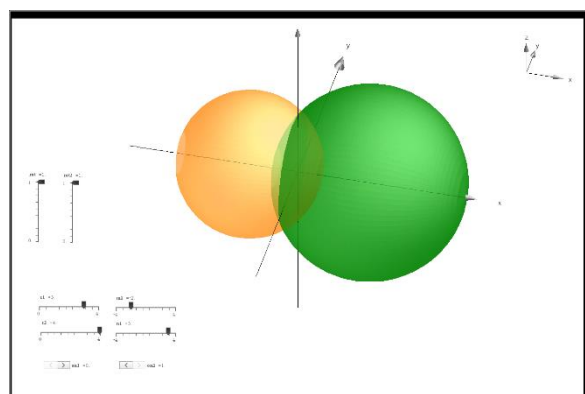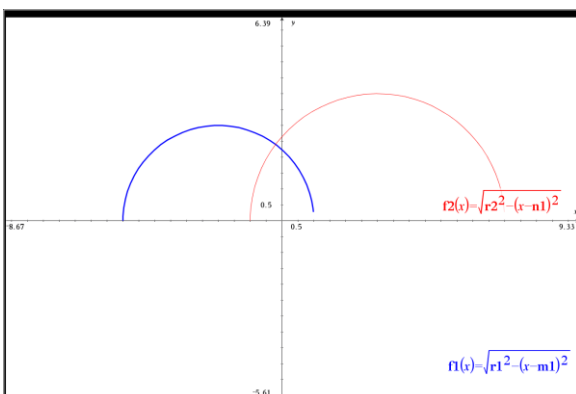
On the graphs page two more sliders that work as a switch have been added. The switch on1 displays or hides the plane containing the circle of intersection of both spheres. Slider on2 displays or hides the right-angled triangle used to calculate the radius of the circle of intersection as an application of the Pythagorean theorem.
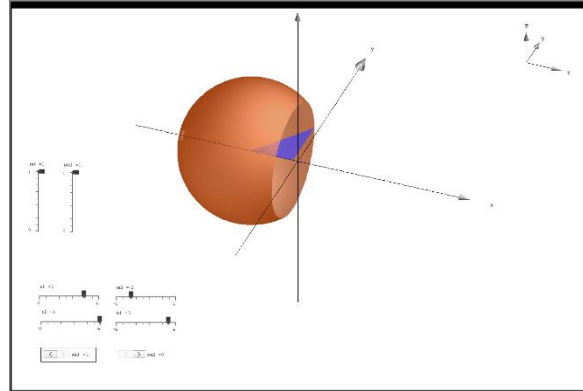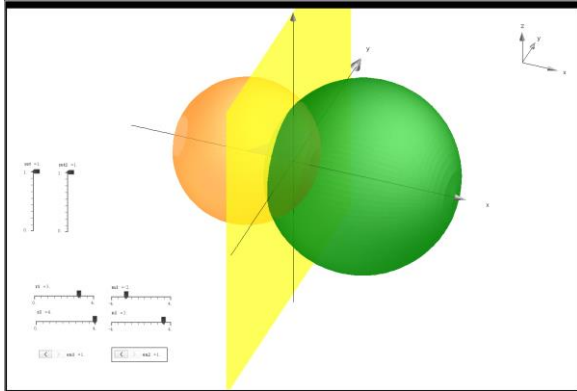
The complete working and all calculations are done using a notes page.


2. calculation:
 – d(B₁,B₂): $|\mathbf{r1-r2}|$ ▸ 1 < norm$(\mathbf{m-n})$ ▸ 5 <$\mathbf{r1+r2}$ ▸ 7
  ⇒ intersecting spheres
 – ∞ points of intersection = circle of points of intersection
 – calculation of the intersecting plane Ω:
 $(x-\mathbf{m1})^2+(y-\mathbf{m2})^2+(z-\mathbf{m3})^2-\mathbf{r1}^2=(x-\mathbf{n1})^2+(y-\mathbf{n2})^2+(z-\mathbf{n3})^2-\mathbf{r2}^2$
  ▸ $x^2+4\cdot x+y^2+z^2-5=x^2-6\cdot x+y^2+z^2-7$

 $(x-\mathbf{m1})^2+(y-\mathbf{m2})^2+(z-\mathbf{m3})^2-\mathbf{r1}^2-(x-\mathbf{n1})^2-(y-\mathbf{n2})^2-(z-\mathbf{n3})^2+\mathbf{r2}^2=0$
  $10\cdot x+2=0$

  ⇒ intersecting plane Ω: $x=\dfrac{-1}{5}$ or $\omega:=\lambda\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+\mu\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}+\begin{bmatrix}\frac{-1}{5}\\0\\0\end{bmatrix}$ ▸ $\begin{bmatrix}\frac{-1}{5}\\\lambda\\\mu\end{bmatrix}$

 – circle = B1 ∩ B2 = B1 ∩ Ω = B2 ∩ Ω   [*see graphs page*]
3. calculation of the centre T of the intersecting circle
 – T is the point of intersection of the line NM with the plane Ω
 – MN $\mathbf{l}:=\delta\cdot(\mathbf{n-m})+\mathbf{n}$ ▸ $\begin{bmatrix}5\cdot\delta+3\\0\\0\end{bmatrix}$

 – solve$(\omega=l,\delta,\lambda,\mu)$ ▸ $\delta=\dfrac{-16}{25}$ and $\lambda=0$ and $\mu=0$

 ⇒ $T = \mathbf{t}:=\mathbf{l}|\delta=\dfrac{-16}{25}$ ▸ $\begin{bmatrix}-1\\\frac{-1}{5}\\0\\0\end{bmatrix}$ or $\omega|\lambda=0.$ and $\mu=0$ ▸ $\begin{bmatrix}-1\\\frac{-1}{5}\\0.\\0\end{bmatrix}$

4. calculation of the radius s of the circle
 – click slider on1 to hide or display ΔTMP

 – d(M,T)= norm$(\mathbf{t-m})$ ▸ $\dfrac{9}{5}$

 – radius of the sphere = $\mathbf{r1}$ ▸ 3

 – Pythagorean theorem in ΔTMP: $|mp|^2=|mt|^2+|tp|^2$

 ⇔ solve$\left(\mathbf{r1}^2=(\text{norm}(\mathbf{t-m}))^2+s^2,s\right)|s\geq0$ ▸ $s=\dfrac{12}{5}$ (s=radius circle ≥ 0)

The TNS file also contains an overview of the relative position in case both spheres have the exact same radius.

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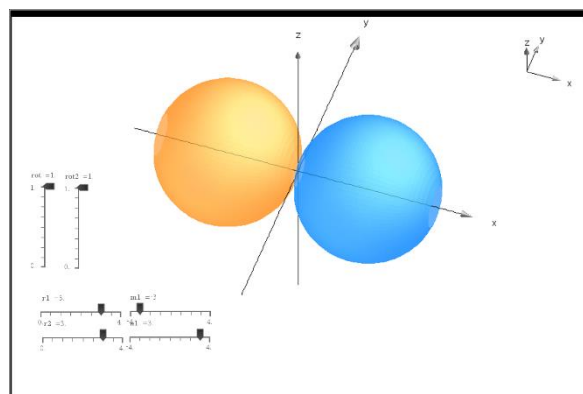
5. HYPERBOLIC FUNCTIONS

5.1 HYPERBOLIC FUNCTIONS and EXPONENTIAL FUNCTIONS

TNS-file: 5.1 HYPERBOLIC 1

Both exponential functions $f(x) = e^x$ and $g(x) = e^{-x}$ are displayed on the graphs page.
Dragging the point a on the X-axis (you can also use the slider instead) creates a scatter plot of the average sum and the average difference of both functions as well as the ratio of these two averages.
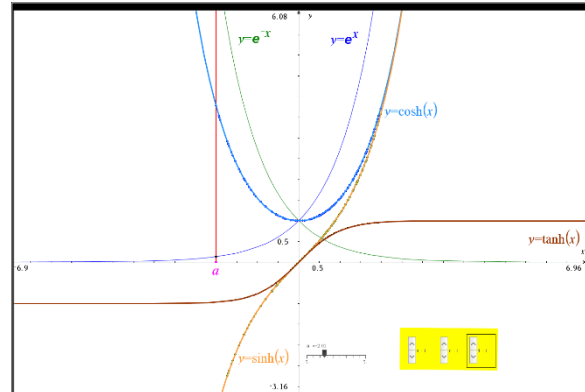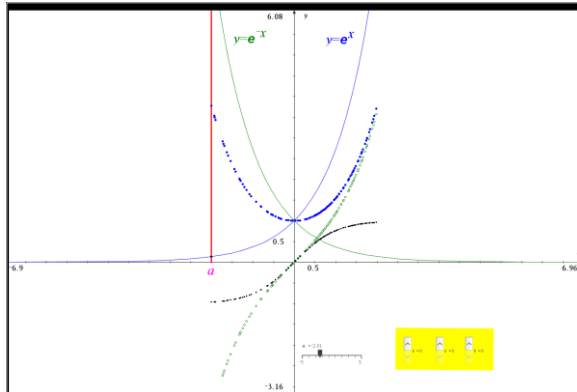
The 3 sliders s, c and t work as a switch to display or hide the hyperbolic functions sinh(x), cosh(x) and tanh(x) respectively.

5.2 HYPERBOLIC FUNCTIONS as TRIGONOMETRIC FUNCTIONS ?

TNS-file: 5.2 HYPERBOLIC 2

To illustrate the similarity between the standard trigonometric functions and the hyperbolic functions a graphs page is used displaying the unit circle and the unit hyperbola $x = \sqrt{1 + y^2}$.

Just as the points (cos t, sin t) form a circle with a unit radius, the points (cosh α , sinh α) form the right half of the equilateral hyperbola.
The hyperbolic functions take a real argument called a hyperbolic angle α . The size of a hyperbolic angle is twice the area of its hyperbolic sector. Half the hyperbolic sector $= \dfrac{\alpha}{2}$ is coloured in orange on the graphs page.

Dragging point P on the hyperbola creates in real time a scatter plot representing the perpendicular projection on the X and Y-axis and thus the corresponding hyperbolic functions.
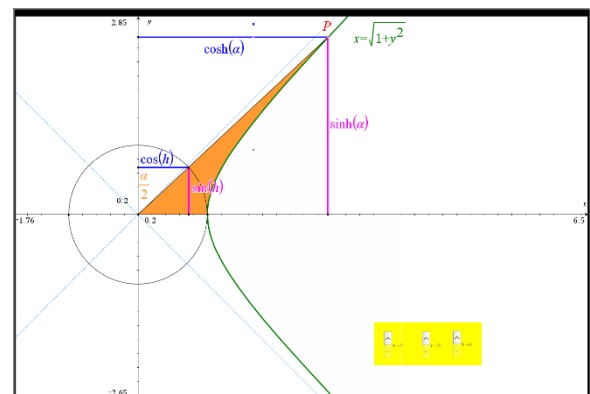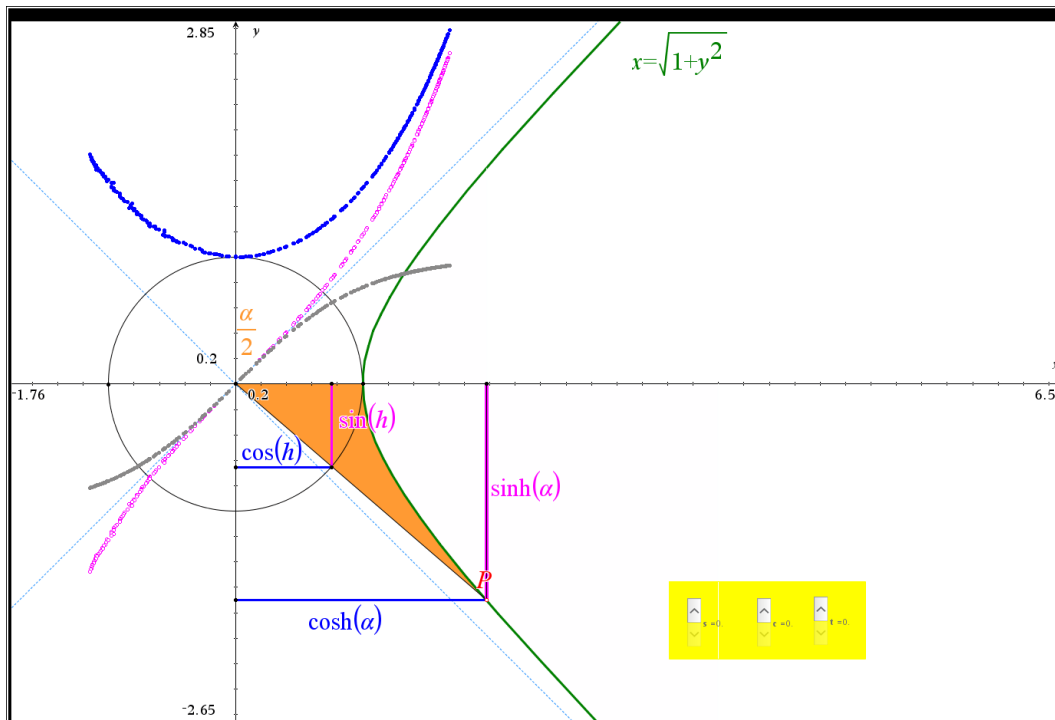
The 3 sliders s, c and t work as a switch with 3 positions.
- 0 = display the scatterplot of sinh(x), cosh(x) or tanh(x) only
- 1 = display the scatterplot and the function
- 2 = display the function only and hide the scatterplot
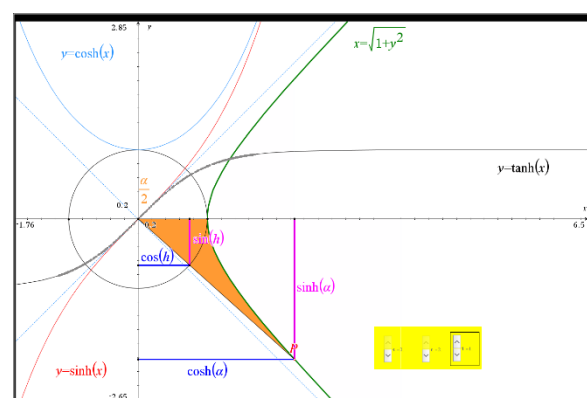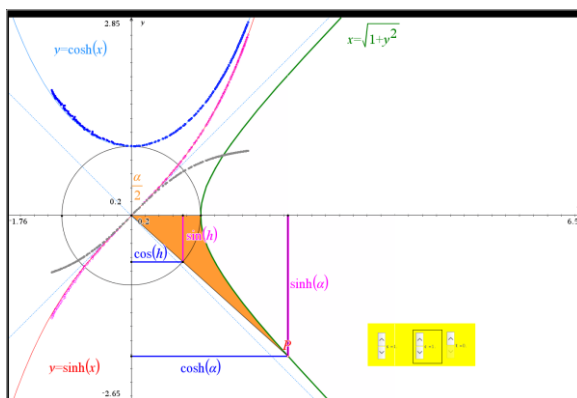
5.3 APPLICATION OF HYPERBOLIC FUNCTIONS: CATENARY

TNS-file: 5.3 CATENARY

We use a picture of a golden chain hanging in a display case as a background for this exploration.

As a first attempt, we try to fit a parabola passing through the vertex and two other points of the chain, the points of suspension.

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Of course, we could do this by calculation but this is not necessary thanks to CAS.

- Drag the vertex of the parabola $y = x^2$ to the correct position on the chain.
- Drag another point of the parabola (not the vertex) to make the parabola as wide as necessary to pass exactly through the two points of suspension.
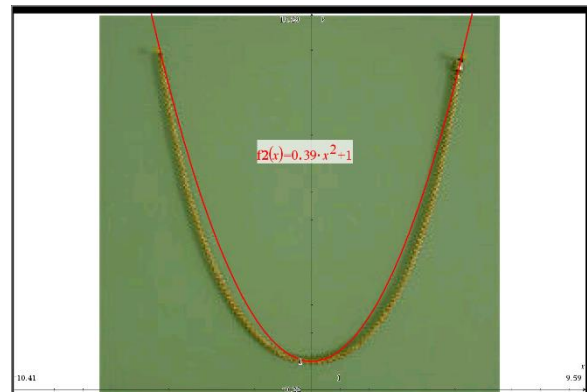
Maybe not so a bad as a first result, but we can do better using a catenary $y = a.\cosh\left(\dfrac{x}{b}\right)$!

- Drag slider a to move the vertex to the correct position on the chain.
- Drag slider b to adjust the width of the catenary.

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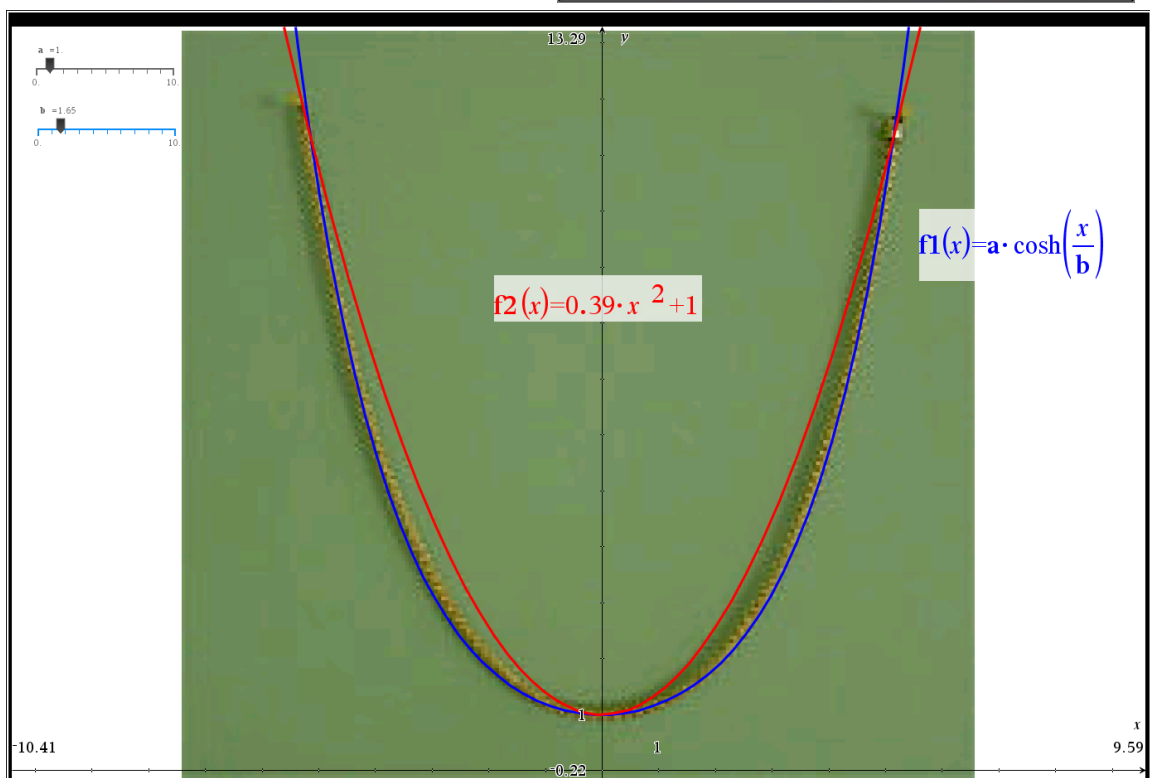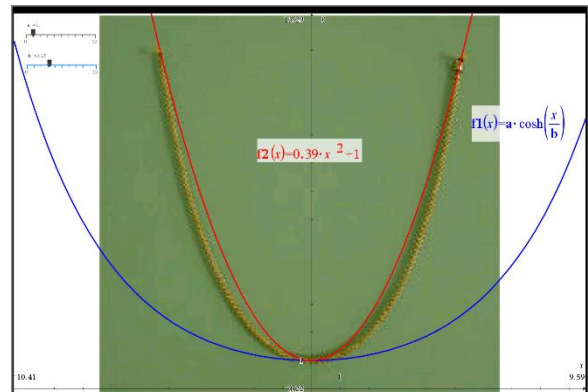
6. DERIVATIVE FUNCTION

6.1 DIFFERENCE QUOTIENT

TNS-file: 6.1 DERIVATIVE SECANT TANGENT LINE

P is a fixed point on the graph of the function. Dragging point b on the X-axis changes the position of the point Q on the graph of the function and therefore the slope of the secant line PQ. This slope as well as the difference quotient are measured and calculated for every new position of point Q.
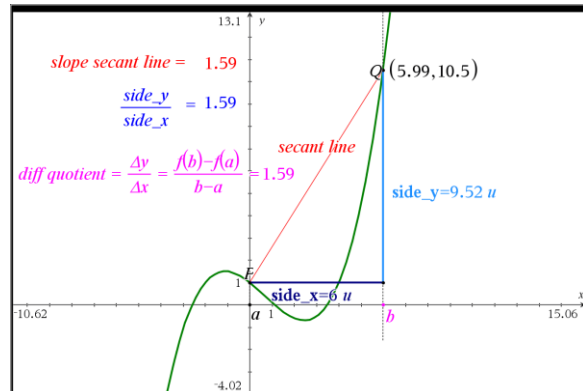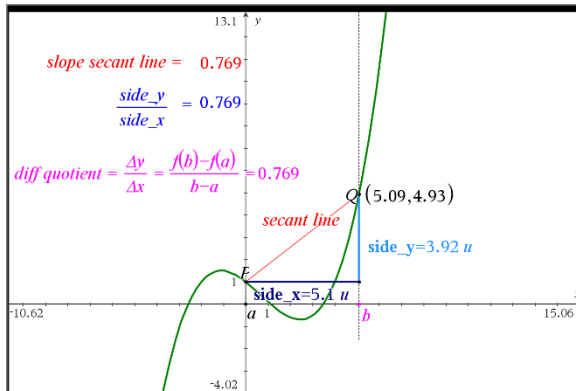
6.2 SECANT LINE – TANGENT LINE - DIFFERENTIABLE

TNS-file: 6.1 DERIVATIVE SECANT TANGENT LINE

The graphs page is divided into two parts. To the left, a fixed punt P on the graph of the function is displayed as well as the tangent line at this point. Another point Q on the same graph is a variable point. Dragging point Q on the graph changes the secant line PQ.
To the right, the constant function represents the slope of the tangent line at P.

Dragging point Q changes the secant line PQ. The constantly changing slope of this secant line is displayed as a scatterplot.
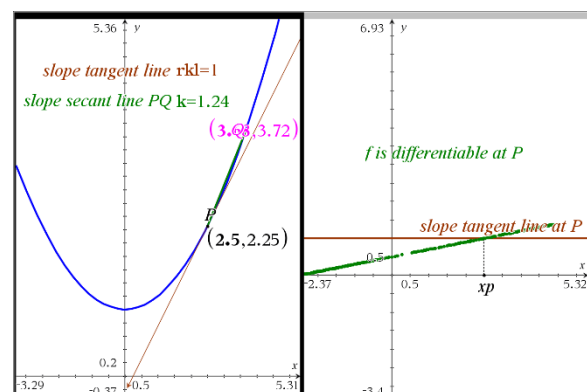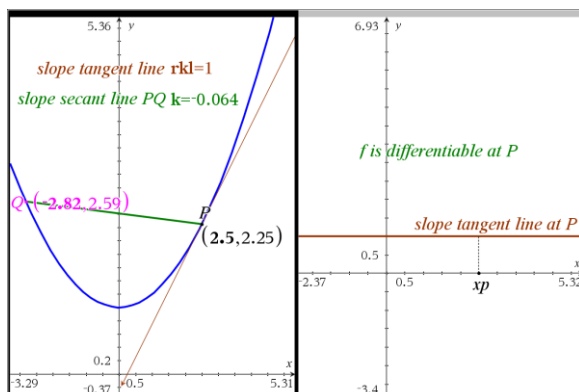
In this case the function is differentiable at P.

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6.3 SECANT LINE – TANGENT LINE – NOT DIFFERENTIABLE

TNS-file: 6.1 DERIVATIVE SECANT TANGENT LINE

Using the same technique an example of a not differentiable function at a point P can be created easily.
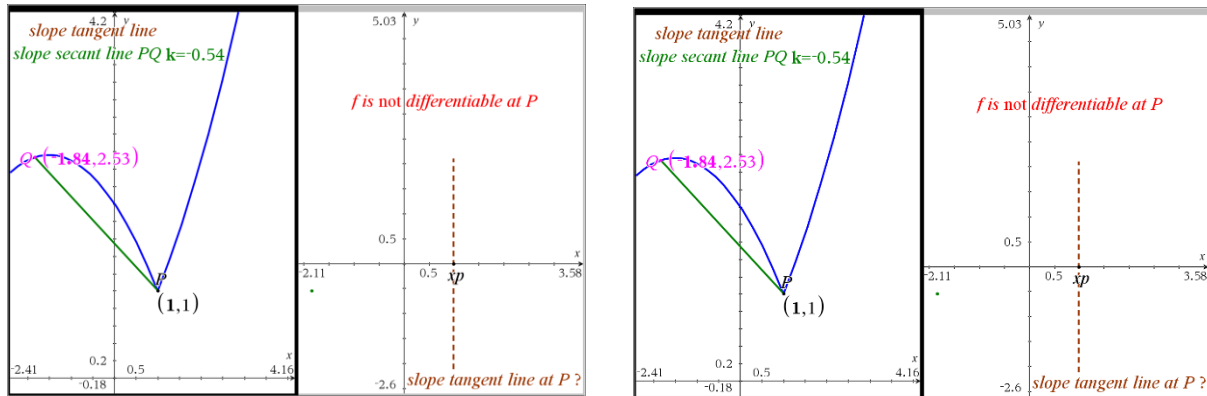
6.4 DERIVATIVE FUNCTION

TNS-file: 6.2 DERIVATIVE FUNCTION

In this problem, P is a variable point on the graph of a quadratic function. The tangent line at P is displayed and it's slope is measured.

Dragging point P on the graph creates a scatter plot representing the constantly changing slope of the tangent line at P or the corresponding derivative function.

Slider v works as a switch to hide or display the derivative function.
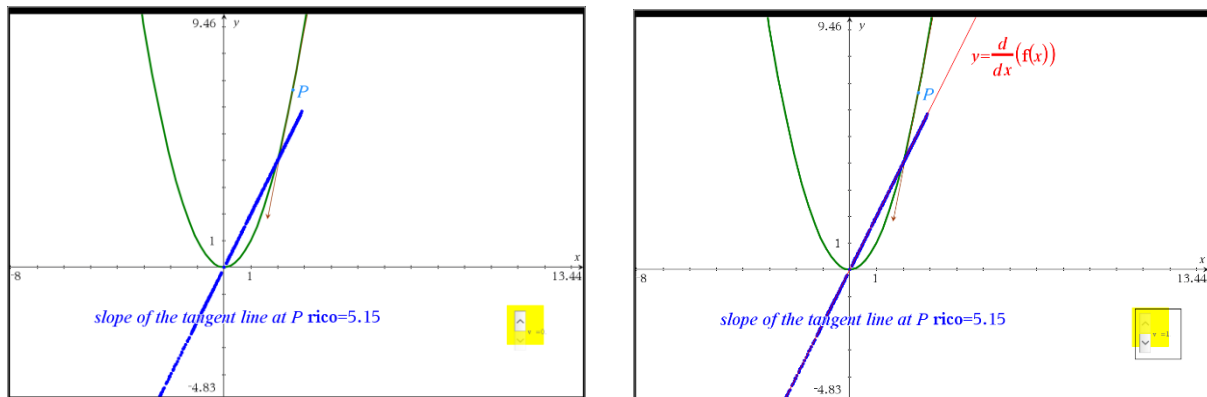
The TNS-file contains also several other examples with different types of functions: linear, trigonometric, etc.

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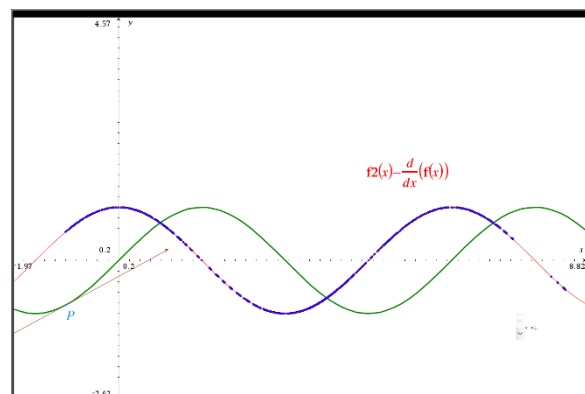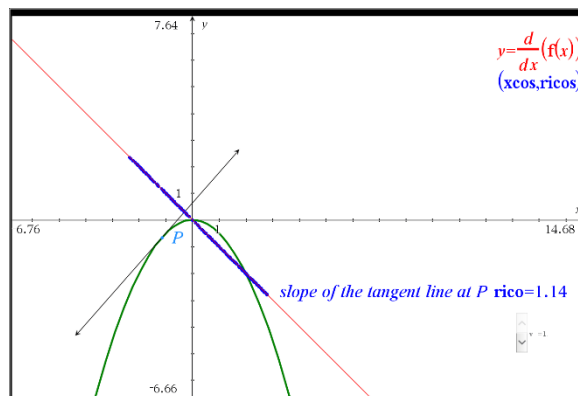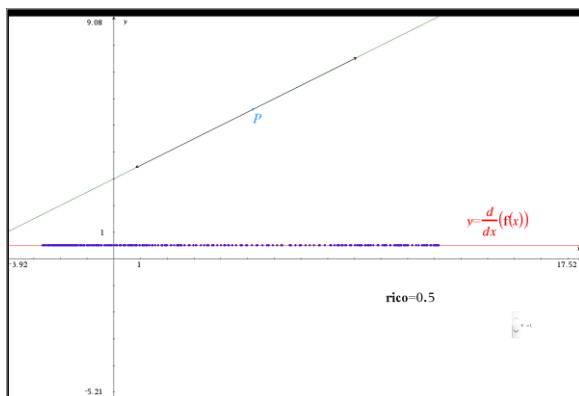
The same approach with the scatter plot representing the slope of a tangent line at a variable point on the graph, can be used to create an example of a not differentiable function at one or more points.

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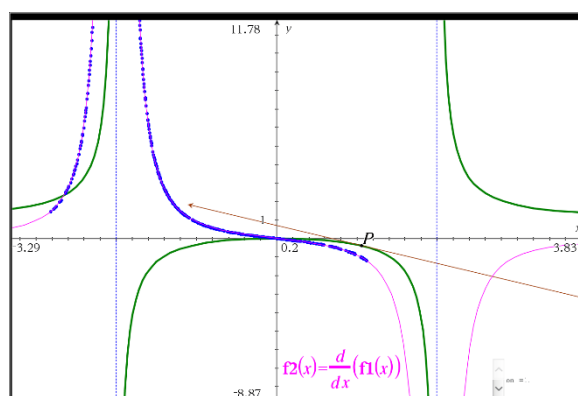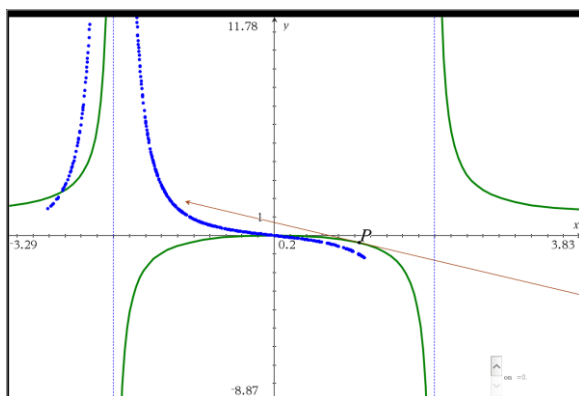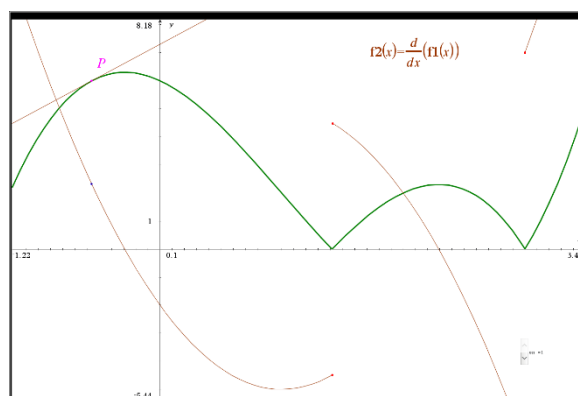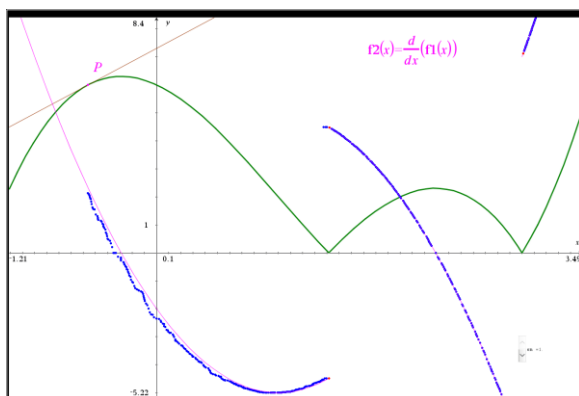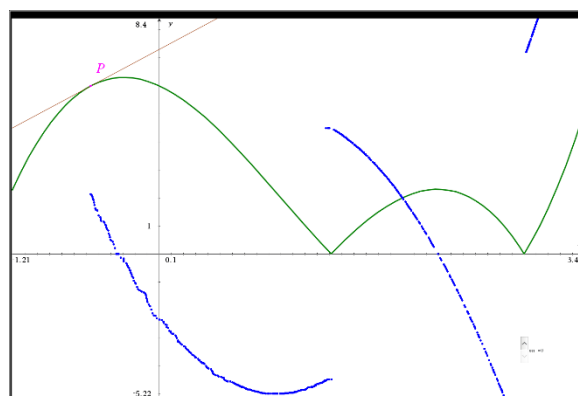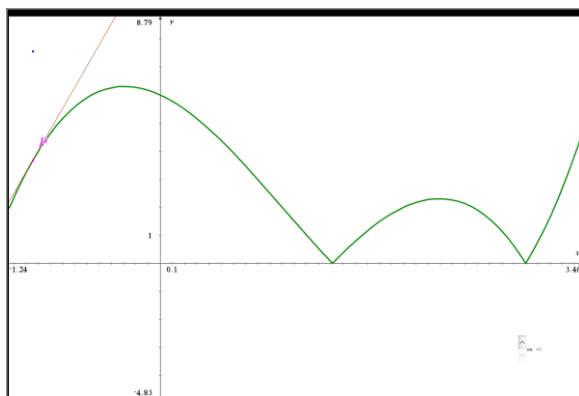
7. VOLUME OF A SOLID OF REVOLUTION

TNS-file: 7.1 SOLID FO REVOLUTION

7.1 CYLINDER

We will use the disk method. As usual, the function is defined on a notes page. The volume of a cylinder is shown and compared to the result using the general formula.

On the first graphs page a slice is drawn using a constant function. The solid is formed by rotating the area between the graph of f(x) and the X-axis and the lines x=a and x=b about the X-axis. To get a cylinder or disk the rotating area must be a rectangle, shown on the first graphs page.

A 3D graphing page is used to visualise the process of rotation about the X-axis. Using the slider rot we can simulate and control the rotation. When the rotation is complete we get a solid of revolution. In this case a disk or cylinder.

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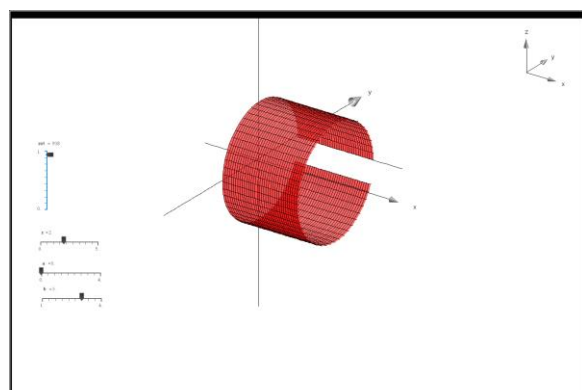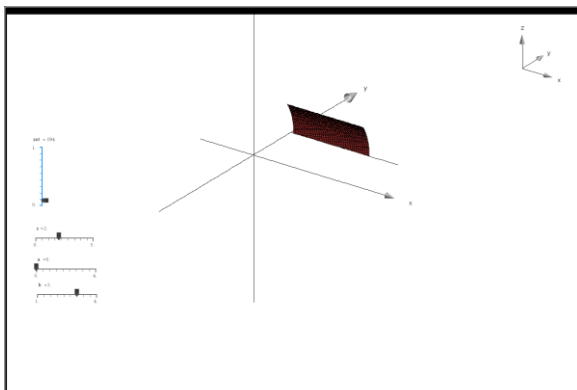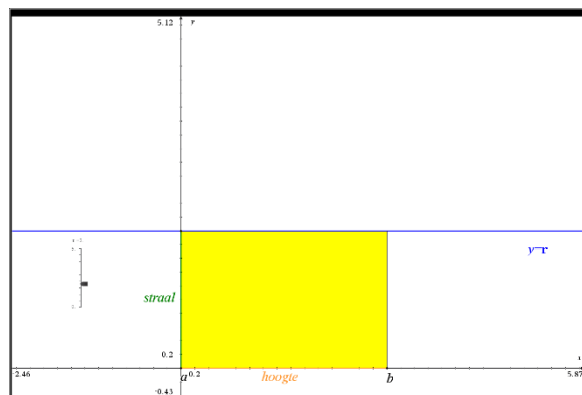
The slider r can be used to adapt the radius of the cylinder. With the two remaining sliders a and b the height of the cylinder can be changed.
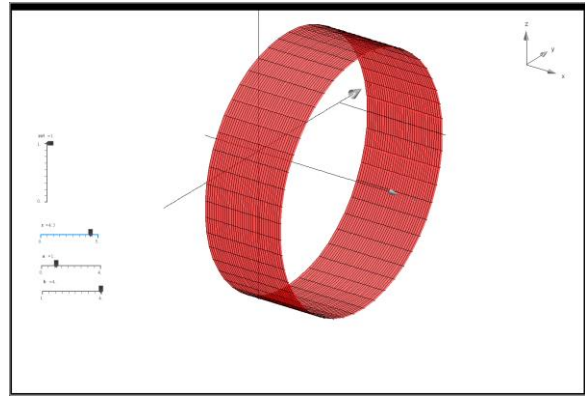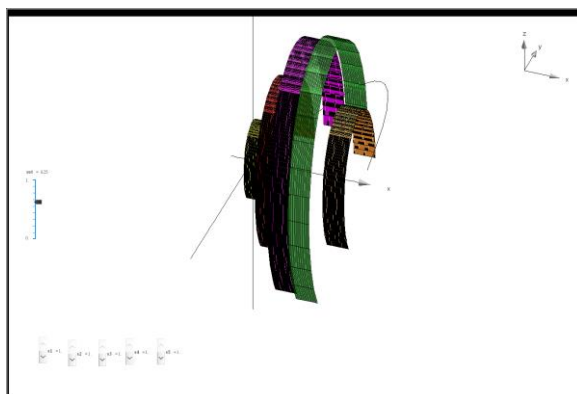

7.2 VOLUME OF A SOLID OF REVOLUTION

In the second problem of the file a possible approximation method for the volume of a solid of revolution is introduced, the disk method. In the example 5 disks are used.

The radius of every disk corresponds to the height of the rotating rectangle, determined by the shape of the graph of the function. The height of the rectangle is the average value of the function in each corresponding interval.

$$f1(x) = \frac{-3 \cdot x \cdot (x+1) \cdot (x-5)}{11} + 1$$
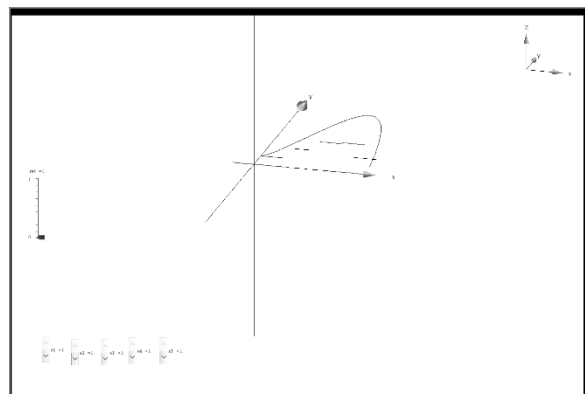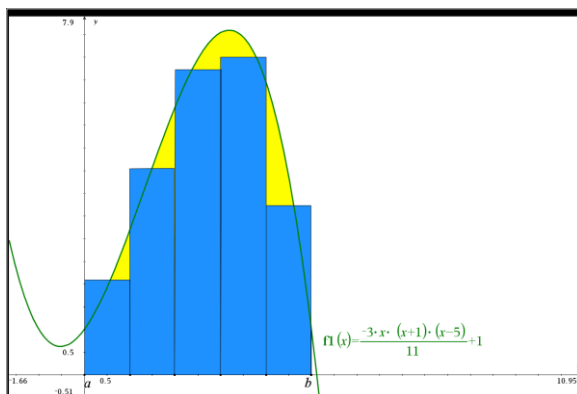
Using 5 rotating rectangles we get 5 disks.
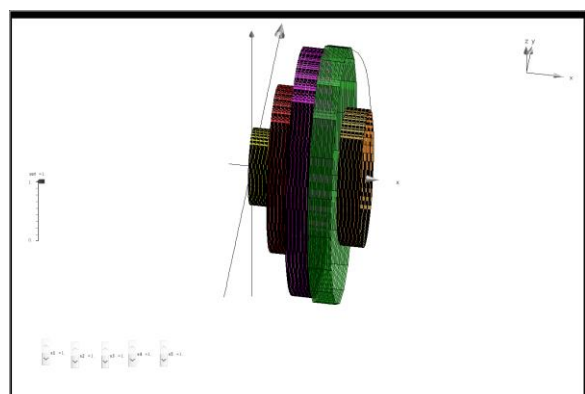
On the 3D graphing page the approximating disks are displayed.
On the second 3D graphing page the five approximating disks can be compared to the solid of revolution.

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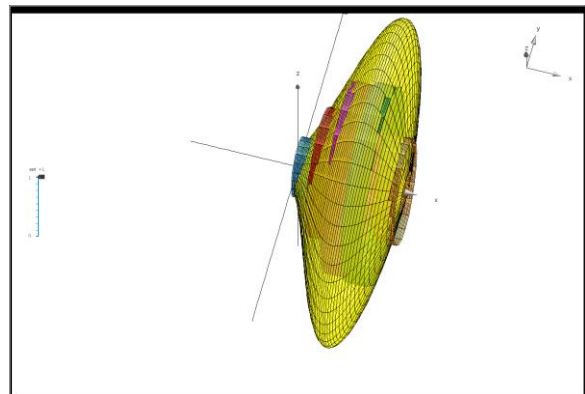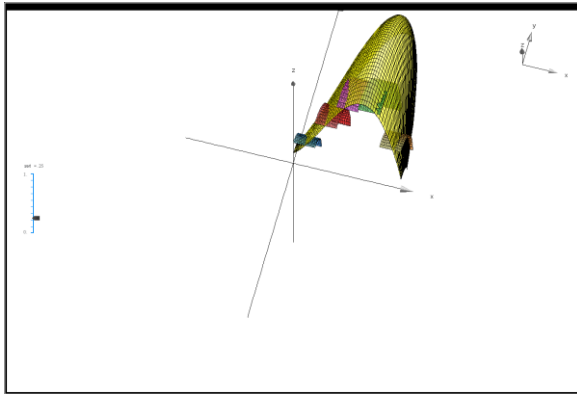
Finally, a notes page is used to compare the volume of the solid of revolution and the approximation of the volume as the sum of the volume of n cylinders.

The slider controls the number n of approximating disks or cylinders. For every new value of n the approximating sum of the disks is recalculated automatically.

7.3 SOME APPLICATIONS

TNS-file: 7.2 VOLUME SOLID FO REVOLUTION

- E.g. 1: torus of revolution

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$f1(x)=4+\sqrt{4-x^2}$

$r\ (radius)$

$(0,4)$

$f2(x)=4-\sqrt{4-x^2}$

$h\ (height)$

0.5

$x=(\text{-}2)$ $\quad 0.5 \quad x=2$

2. see graphs page 2
3. volume of the torus

$$\text{volume torus}(4,2) = \int_{-2}^{2} \left(\pi \cdot \left((f1(x))^2 - (f2(x))^2 \right) \right) dx \; \blacktriangleright \; 32 \cdot \pi^2$$

4. general solution:

$$f(x):=h+\sqrt{r^2-x^2}$$

$$g(x):=h-\sqrt{r^2-x^2}$$

$$\text{volume torus}(h,r) = \int_{-r}^{r} \left(\pi \cdot \left((f(x))^2 - (g(x))^2 \right) \right) dx \,|\, r>0 \; \blacktriangleright \; 2 \cdot h \cdot \pi^2 \cdot r^2$$
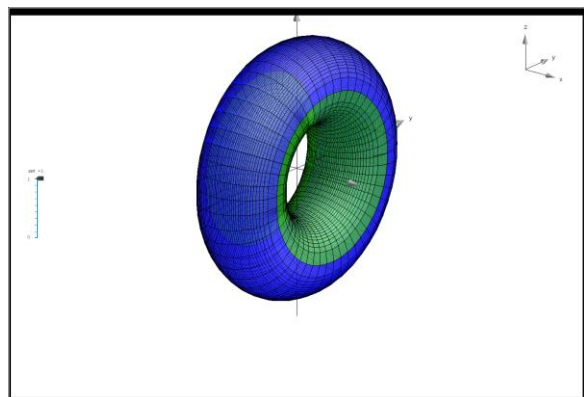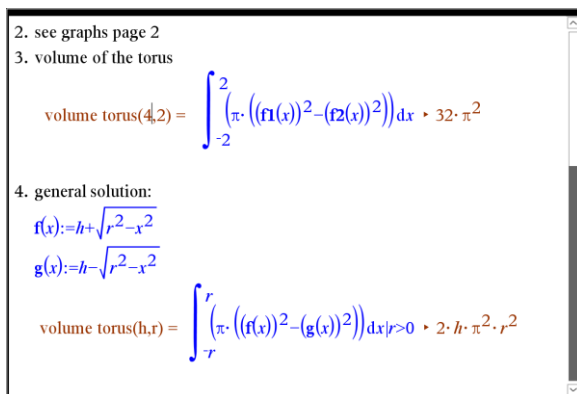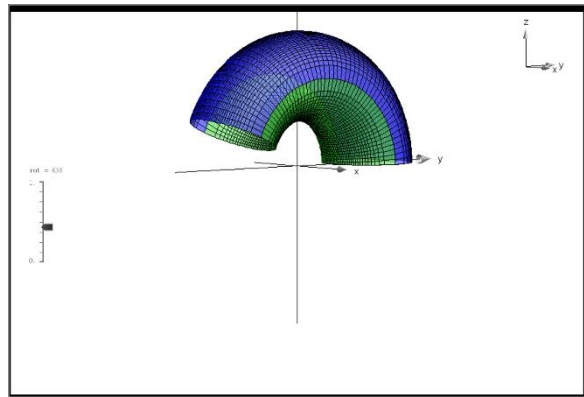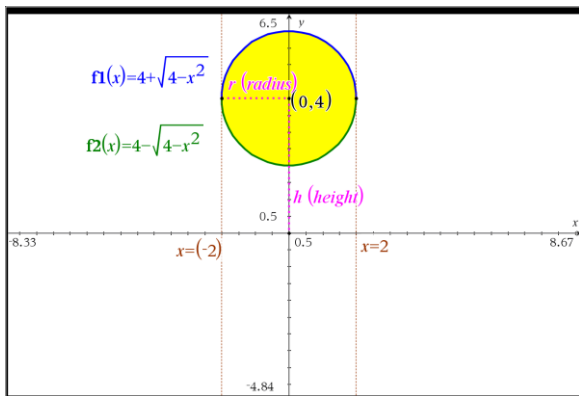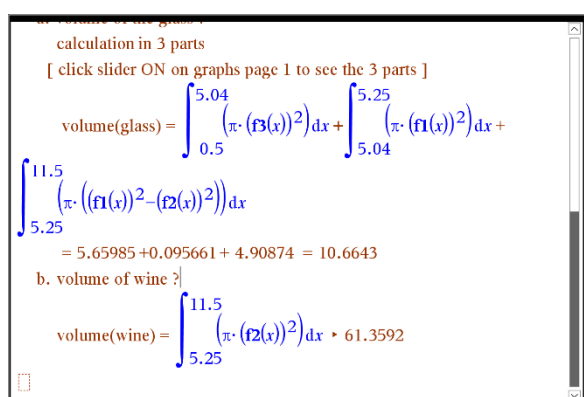

- E.g. 2: wine glass as a solid of revolution

$f3(x)=\frac{1}{x}$

$f1(x)=\sqrt{x-5}$

$f2(x)=\sqrt{x-5.25}$

$x=0.5$ $\qquad x=11.5$

$f3(x)=\frac{1}{x}$

$f1(x)=\sqrt{x-5}$

$f2(x)=\sqrt{x-5.25}$

$x=5$ $\quad x=5.25$

$x=0.5$

$x=5.04$

$x=11.5$

calculation in 3 parts
[click slider ON on graphs page 1 to see the 3 parts]

$$\text{volume(glass)} = \int_{0.5}^{5.04} \left(\pi \cdot (f3(x))^2 \right) dx + \int_{5.04}^{5.25} \left(\pi \cdot (f1(x))^2 \right) dx +$$

$$\int_{5.25}^{11.5} \left(\pi \cdot \left((f1(x))^2 - (f2(x))^2 \right) \right) dx$$

$$= 5.65985 + 0.095661 + 4.90874 = 10.6643$$

b. volume of wine ?

$$\text{volume(wine)} = \int_{5.25}^{11.5} \left(\pi \cdot (f2(x))^2 \right) dx \; \blacktriangleright \; 61.3592$$
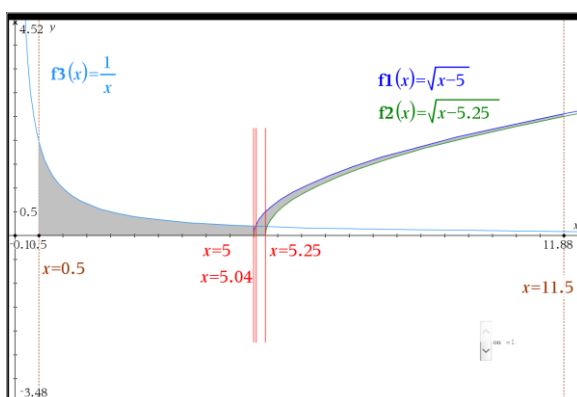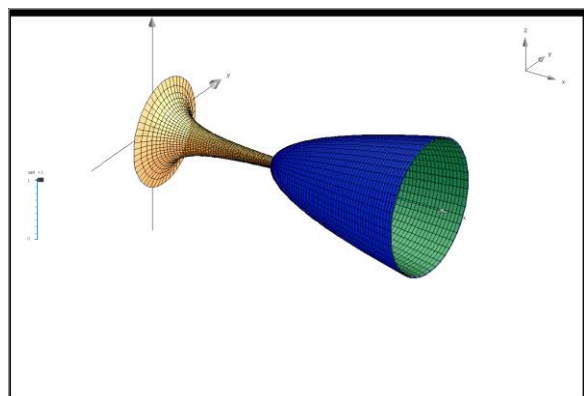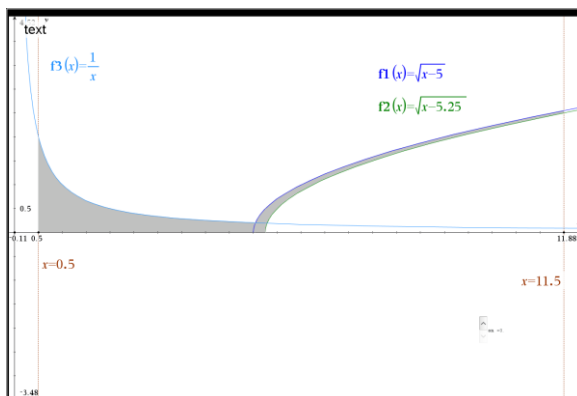
Using CAS, the calculation of the volume of these solids of revolution is easy enough.

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