Applied Mathematics – a tribute to Leonhard Euler’s 315th birthday

Hubert Langlotz

An Euler song

- https://www.youtube.com/watch?app=desktop&v=8KJtazJMyl0

"Read Euler, read Euler. He is the master of us all."

–*Laplace*

Leonhard Euler - one of the most productive mathematicians

Euler and Number Theory
Euler and Logarithms
Euler and Infinite Series
Euler and Analytic Number Theory
Euler and Complex Variables
Euler and Algebra
Euler and Geometry
Euler and Combinatorics . . .

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{\ddots}}}}$$

Some examples of the occurence of „e“

$$\lim_{n \to \infty}\left(\left(1+\frac{1}{n}\right)^n\right) \qquad e$$

$$\sum_{k=0}^{1000}\left(\frac{1}{k!}\right) \qquad 2.71828$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + - \ldots = \frac{1}{e}\,.$$

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How we teach in school “e“ – some examples

- Interest rates → steady Interest rates
- Start with 1€ with an Interest rate of 100% (☺)
- Interest once a year 1€+ 100%*1€ = 2€
- Interest twice a year 1€(1+1/2)^2 = 2,25€
- ...
- Interest n-times a year 1€(1+1/n)^n ≈ ?

$$\lim_{n \to \infty}\left(\left(1+\frac{1}{n}\right)^{n}\right) \qquad e$$

f'(x) = f(x) mit f(x) = a^x

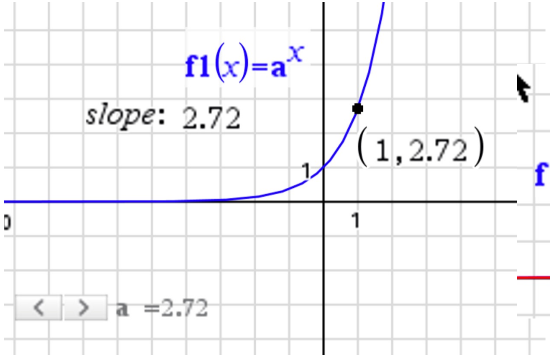
$$f'(x)=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x\cdot a^h-a^x}{h}=a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}=a^x\cdot\lim_{h\to 0}\frac{a^{0+h}-a^0}{h}=a^x\cdot f'(0)$$

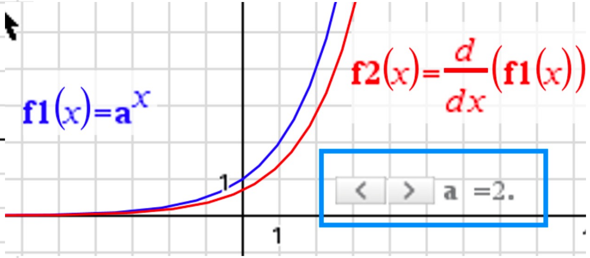
Investigate the value of a by evaluating $m(a) = \lim_{h\to 0}\frac{a^h-1}{h} = 1$

$$\text{seq}\left(\lim_{h\to 0}\left(\frac{a^h-1}{h}\right),a,2.7,2.8,0.01\right)$$

$\{0.993252,0.996949,1.00063,1.0043,1.0079$ ▸

$$\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)|a=e \qquad 1$$

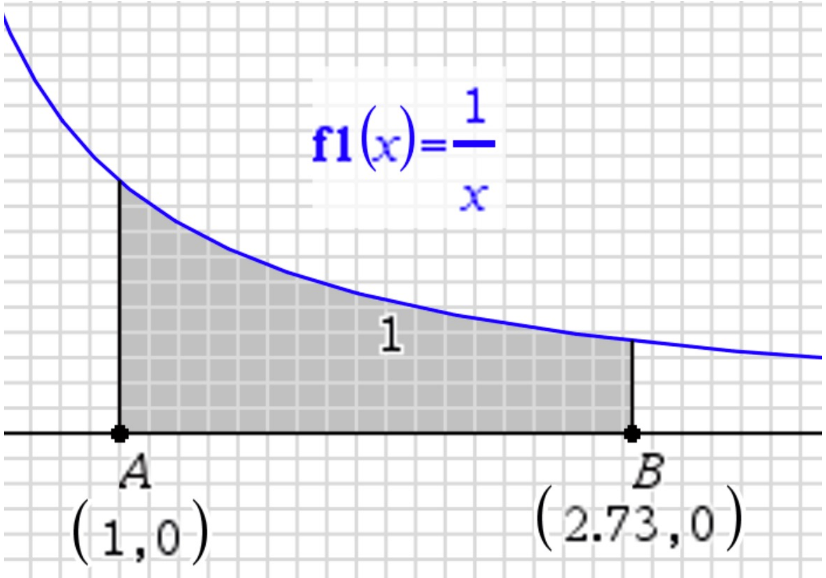




$$F_1(x) = \int_1^x \frac{1}{t} d(t) = 1$$

$$\int_1^x \frac{1}{t}\, dt = 1 \qquad \ln(|x|) = 1$$

$$\text{solve}(\ln(|x|) = 1, x) \qquad x = -e \text{ or } x = e$$



Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \left(= \sum_{k=0}^{\infty} \frac{x^k}{k!} \right)$$

$\text{taylor}\left(e^x, x, 5\right)$	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$
$\text{taylor}\left(e^x, x, 5\right) \vert x=1$	$\frac{163}{60}$
$\text{taylor}\left(e^x, x, 5\right) \vert x=1$	2.71667

$e^{i\cdot\pi} = -1 \;\; or \;\; e^{i\cdot\pi} + 1 = 0$

- Background: Taylor Series
- with $i^2 = -1 \; and \; i^4 = 1 \ldots$

we find $e^{i\cdot x} = \cos(x) + i\cdot\sin(x)$

With x $=\pi$ we find

$e^{i\cdot\pi} = -1$

taylor$(e^{x},x,5)$	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$
taylor$(\sin(x),x,5)$	$x-\frac{x^3}{6}+\frac{x^5}{120}$
taylor$(\cos(x),x,5)$	$1-\frac{x^2}{2}+\frac{x^4}{24}$

1. Python example

1.1 | 1.2 | 1.3 | *Eulerfolge1 RAD

*Eulerfolge1.py 1/56

```
from math import *
from ti_system import *

def euler(n):
  a = 1
  #Listen zur grafischen Darstellung
  ilist = []; alist=[]
  for i in range(1, n + 1):
    #Berechnung des i−ten Folgengliedes
    a = (1 + 1 /  i) ** i
    ilist.append(i)
```

Some more unexpected occurences of „e“

Example 1

Fixpointfree permutations – derangement – (rencontre)

Hat check problem: If n men have their hats randomly returned,

what is the probability that at least one of the men winds up with his own hat?

$$P(n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \ldots + (-1)^{n+1} \cdot \frac{1}{n!} \approx 1 - \frac{1}{e} \approx 0{,}632$$

How to teach fixpointfree permutations

"Run out of money at the bar ... let your friend pay!"

Greg Gutfeld Men's Health magazine

- Have him shuffle two decks of cards and lay them side by side. Explain that you always take one card from each deck from the top at the same time and bet that a pair of identical cards eventually appear.
- We encounter this paradoxical process in a variety of scenes (secretary problem, mixed-up letters problem...).

$grundmenge(n):=\text{seq}(k,k,1,n)$	*Fertig*
$perm(n):=\text{randSamp}(grundmenge(n),n,1)$	*Fertig*
$perm(5)$	$\{5,4,2,1,3\}$
$perm(5)$	$\{3,2,5,1,4\}$
$\{5,4,2,1,3\}-\{3,2,5,1,4\}$	$\{2,2,^-3,0,^-1\}$
$\text{countIf}(\{2,2,^-3,0,^-1\},0)$	1
$\text{seq}(\text{countIf}(perm(5)-perm(5),0),k,1,100)$	
$\{3,1,1,1,2,1,1,1,0,0,0,1,2,0,0,0,1,0,0,2,1,0,0,0,1,1,1,1,0,1,0,0,1,0$ ▸	
$\frac{\text{countIf}(\text{seq}(\text{countIf}(perm(5)-perm(5),0),k,1,1000),?>0)}{1000}$	0.637

… see material

Fixedpointfreepermutations.pdf

Example 2

The following appeared as a Putnam examination problem [1]:

If numbers are randomly selected from the interval [0, 1], what is the expected number of selections necessary until the sum of the chosen numbers first exceeds 1? The answer is e.

An elementary proof of this is given in [2].

[1] L.E. Bush, The William Lowell Putnam Mathematical Competition,

Amer. Math.Monthly 68 (1961), 18-33.

[2] H.S. Shultz, An expected value problem, Two-Year College Math. J. 10 (1979), 179.

Two simulations

A wurf	B summe	C	D liste
=rand(10)	=cumulati		=capture(
0.78306	0.735197	2.	4
0.728228	1.08639		2
0.517898	1.81557	2.73684	2
0.712745	2.21922		4

$zz:=\text{countif}(\text{summe},?<1)+1+\frac{\text{rand}()}{10^{10}}$

eulerzufall2.py 1/22

```
from random import *

def eulerzufall(n):
  #Gesamtzähler für die Zufallsversuche:
  m = 0
  for i in range(n):
    #z: Summe der Zufallszahlen
    #j: Zähler der Zufallszahlen bis die Summe
    z = 0; j = 0
    while z < 1:
      z += random()
```

Some more ideas

- **Raisin problem**
- **Secretary problem**
- Stirling‘s formula
- Gaussian function
- Minimum of f(x) = x^x
- ...

Raisin Problem

You have a dough for 100 rolls and 100 raisins. Carefully knead the raisins into the dough.

How many rolls contain at least one raisin?

	A rosine	B brötchen	C anzahl	D
=	=randint(1	=seq(k,k,1		
1	23	1	1	61
2	79	2	0	
3	100	3	2	
4	83	4	1	
5	85	5	1	

C3 =countif(rosine,b3)

$\text{binomCdf}\left(100,\frac{1}{100},1,100\right) \blacktriangleright 0.633968$

Mindestens eine Rosine pro Feld:

$P(A) = 1-P(0) = 1-(0.99)^{100} \blacktriangleright 0.633968$

```
def rosinen(n):
    r = [randint(1,n) for i in range(n)]
    z = 0
    for j in range(n):
        k = 0
        for i in range(n):
            if r[i] == j: k = 1; break
        z += k
    return z / n
```

Unexpected occurences of the number “e“

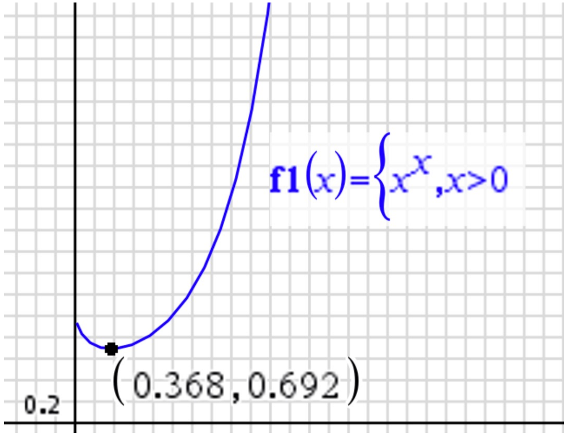
Example 3. The “Secretary Problem” concerns an employer who is about to interview n applicants for a secretarial position. At the end of each interview he must decide whether or not this is the applicant he wishes to hire. Should he pass over an interviewee, this person cannot be hired thereafter. If he gets to the last applicant, this person gets the job by default. The goal is to maximize the probability that the person hired is the one most qualified. His strategy will be to decide upon a number $k < n$, to interview the first k applicants, and then to continue interviewing until an applicant more qualified than each of those first k is found. As seen in [**3**], the probability of hiring the most qualified applicant is greatest when k/n is approximately $1/e$. Moreover, this number, $1/e$, is in fact the approximate maximum probability. For example, if there are $n = 30$ applicants, the employer should interview 11 (which is approximately $30/e$) and then select the first thereafter who is more qualified than all of the first eleven. The probability of obtaining the most qualified applicant is approximately $1/e$.

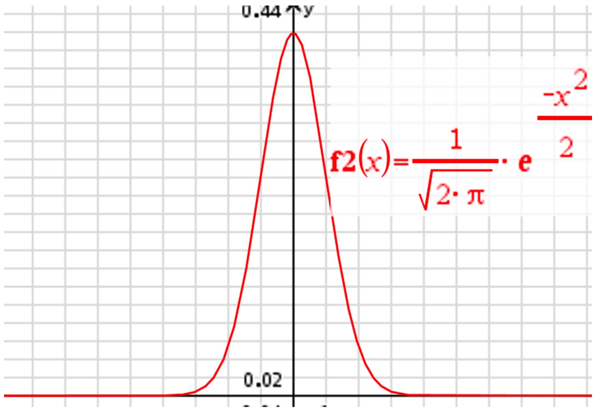
https://www.maa.org/sites/default/files/268977052054.pdf

Example 4. With each purchase, a certain fast-food restaurant chain gives away a coin with a picture of a state capitol on it. The object is to collect the entire set of 50 coins. Question: After 50 purchases, what fraction of the set of 50 coins would one expect to have accumulated? In [**5**] it is shown that this fraction is $1 - \left(1 - \frac{1}{50}\right)^{50}$, which is approximately $1 - 1/e$.

Example 5. A sequence of numbers, $x_1, x_2, x_3, \ldots$, is generated randomly from the interval $[0, 1]$. The process is continued as long as the sequence is monotonically increasing or monotonically decreasing. What is the expected length of the monotonic sequence? For example, for the sequence beginning .91, .7896, .20132, .41, the length of the monotonic sequence is three. For the sequence beginning .134, .15, .3546, .75, .895, .276, the length of the monotonic sequence is five.

```
def eulerzufall(n):
    m = 0
    for i in range(n):
        z0 = 0; z = random()
        j = 1
        while z0 < z:
            z0 = z
            z = random()
            j += 1
        m += j
    return m / n
```





$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \to \infty.$$

$\frac{d}{dx}(f1(x))$	$\{(\ln(x)+1)\cdot x^x, x>0$
$\text{solve}((\ln(x)+1)\cdot x^x=0,x)$	$x=0.367879$
$\frac{1}{e}$	0.367879
$f1\left(\frac{1}{e}\right)$	$e^{-e^{-1}}$

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{\ddots}}}}$$

Further sources

- https://math.stackexchange.com/questions/1646042/what-are-some-surprising-appearances-of-e
- https://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/mccartin.pdf
- https://www.youtube.com/watch?app=desktop&v=8KJtazJMyl0
- https://knowallwith.wordpress.com/2017/04/03/a-formula-which-can-get-first-18-trillion-trillion-digits-of-the-constant-e-which-is-truly-amazing/

Any questions: simply shoot me an email: hlanglotz@gmx.de

$$\frac{878}{323} \qquad 2.71827$$

Top ln(e^{10}) Reasons Why e Is Better than π

10) e is easier to spell than π.
9) $\pi \approx 3.14$ while $e \approx 2.718281828459045$.
8) The character for e can be found on a keyboard, but π sure can't.
7) Everybody fights for a piece of the pie.
6) $\ln(\pi)$ is a really nasty number, but $\ln(e) = 1$.
5) e is used in calculus while π is used in baby geometry.
4) 'e' is the most commonly picked vowel in Wheel of Fortune.
3) e stands for Euler's Number; π doesn't stand for squat.
2) You don't need to know Greek to be able to use e.
1) You can't confuse e with a food product.

https://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/mccartin.pdf