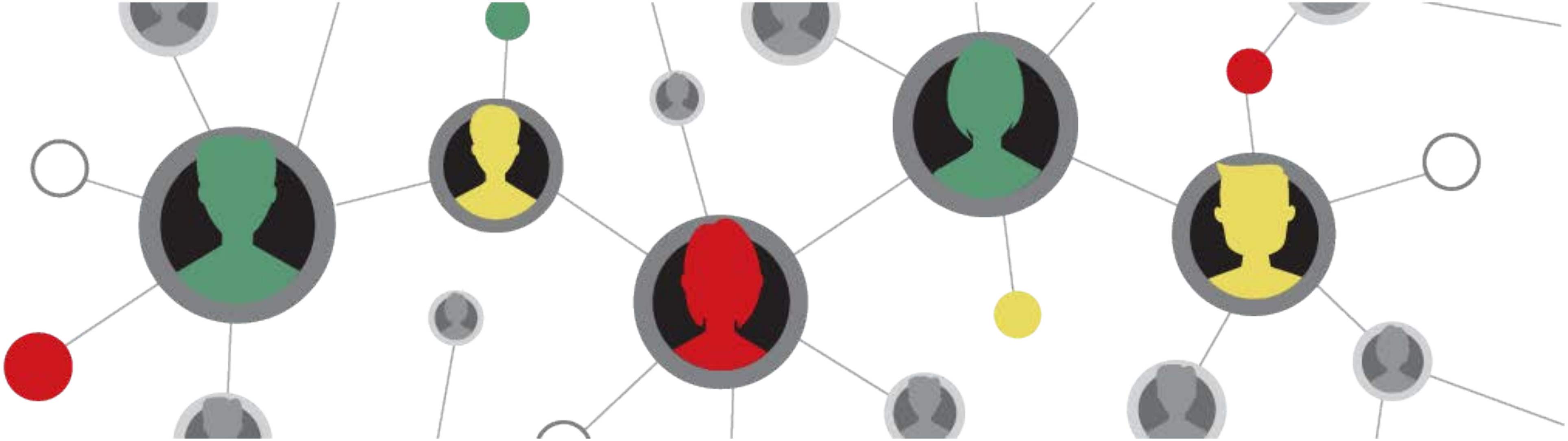
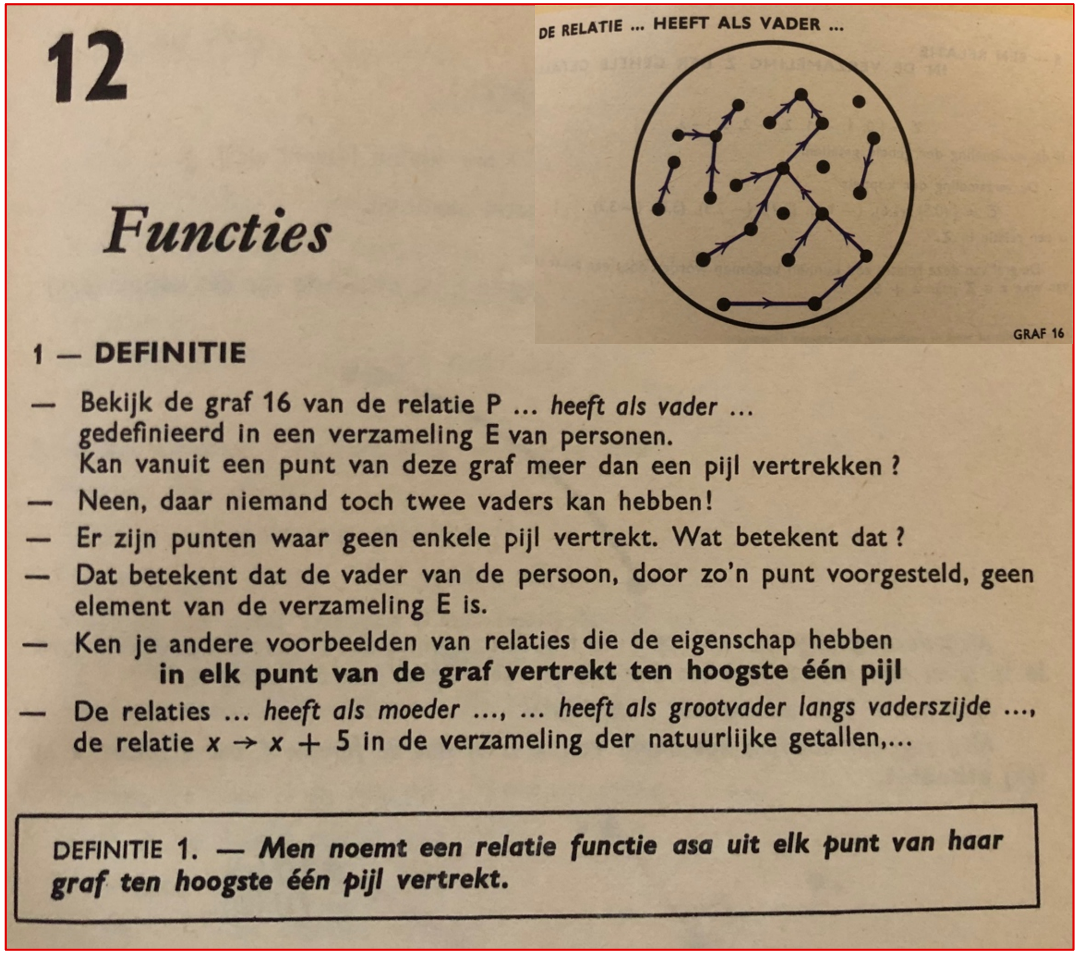
# Reeële functies wat anders bekeken

*Koen Stulens*



# Reële functies wat anders bekeken

## 1965 – *Moderne Wiskunde – Papy*

12

*Functies*

DE RELATIE ... HEEFT ALS VADER ...

GRAF 16

**1 — DEFINITIE**

— Bekijk de graf 16 van de relatie P ... *heeft als vader* ... gedefinieerd in een verzameling E van personen.
Kan vanuit een punt van deze graf meer dan een pijl vertrekken ?

— Neen, daar niemand toch twee vaders kan hebben!

— Er zijn punten waar geen enkele pijl vertrekt. Wat betekent dat ?

— Dat betekent dat de vader van de persoon, door zo'n punt voorgesteld, geen element van de verzameling E is.

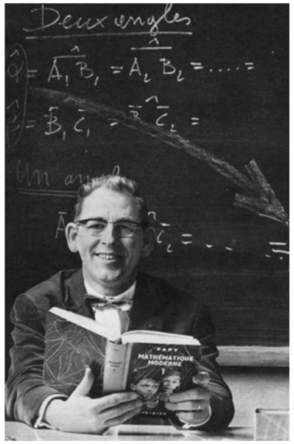
— Ken je andere voorbeelden van relaties die de eigenschap hebben
**in elk punt van de graf vertrekt ten hoogste één pijl**

— De relaties ... *heeft als moeder* ..., ... *heeft als grootvader langs vaderszijde* ..., de relatie $x \to x + 5$ in de verzameling der natuurlijke getallen,...

DEFINITIE 1. — ***Men noemt een relatie functie asa uit elk punt van haar graf ten hoogste één pijl vertrekt.***

# Reële functies wat anders bekeken

## 1965 – *Moderne Wiskunde* – Papy




DEFINITIE 1. — ***Men noemt een relatie functie asa uit elk punt van haar graf ten hoogste één pijl vertrekt.***

*Als* **x** *de oorsprong van een koppel van* **f** *is, dan zegt men dat de functie* **f gedefinieerd is in x** *en men noteert* **f(x)** *het uiteinde van het koppel met oorsprong* **x**.

*Men noemt* **f(x)** ***de waarde van* f *in* x.**

*Men zegt ook dat* **f(x)** ***het beeld van* x** *is of dat de functie* **f**, *het element* **x** *op* **f(x) afbeeldt.**

**2.** Stellen we door $f$ de functie $x \to x + 5$ voor.
Bereken $f(2)$, $f(17)$.
Welk is het beeld van nul door $f$?
Op welk getal beeldt $f$ het getal 7 af?
Welk is het getal waarvan 8 het beeld door $f$ is?
Men zegt je dat $f(a) = 87$. Bereken $a$.

**3.** Duiden we door $d$ de functie ... *heeft als dubbel* ... aan.
Welk is het beeld van 8 door $d$?
Welk getal heeft 18 als beeld?
Als $d(u) = 100$, bereken dan $u$.
Op welk getal beeldt de functie $d$ nul af?
Hoeveel is $x$ als $d(x) = x$?

# Reële functies wat anders bekeken

## 1983 – *Algebra Deel 1 – LUC, Ooms*

**3. Relaties:**

**Definitie:** Een deelverzameling $R \subset A \times B$ van het cartesisch product $A \times B$ noemt men een relatie van A naar B. Men schrijft meestal $aRb$ in plaats van $(a,b) \in R$.

Vb: $R = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} \text{ zodat } y = qx\} \subset \mathbb{N} \times \mathbb{N}$

R is de relatie „is een deler van". M.a.w. $(x,y) \in R$ betekent $x$ deelt $y$.

**4. Afbeeldingen:**

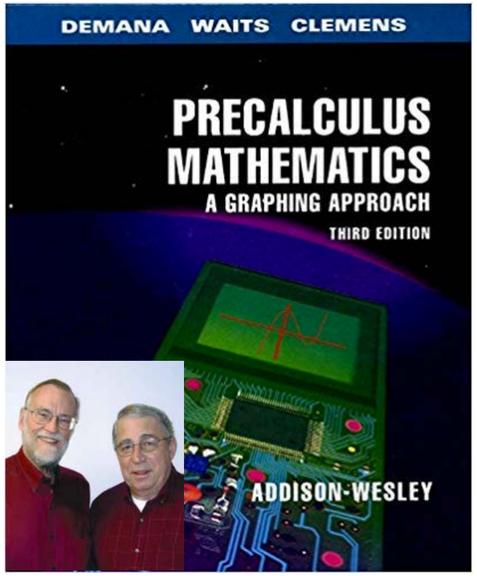
**Definitie:** f heet een afbeelding (of functie) van een verzameling X in een verzameling Y (notatie: $f: X \to Y$) als f een relatie is van X in Y zodat voor elke $x \in X$ één en slechts één element $y \in Y$ overeenkomt waarvoor geldt $(x,y) \in f$.

Anders uitgedrukt: f is een relatie van X in Y,

Men schrijft $y = f(x)$ in plaats van $(x,y) \in f$, en men noemt $f(x)$ het beeld van $x$ onder $f$. Maak goed onderscheid tussen $f(x)$ en $f$: $f$ is een functie, terwijl $f(x)$ een element is van Y. Twee afbeeldingen $f, g$ van X in Y zijn gelijk als $f(x) = g(x)$ voor elke $x \in X$.

# Reeële functies wat anders bekeken

## 1993 – *Precalculus Mathematics: A Graphing Approach* – B Waits




**Definition Function**

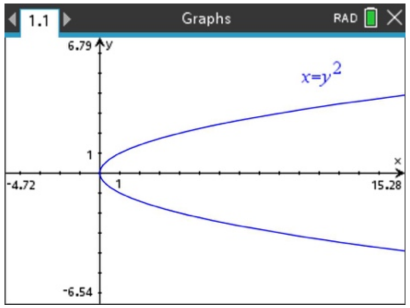
A **function** from a set $D$ to a set $R$ is a rule that assigns a unique element in $R$ to each element in $D$.

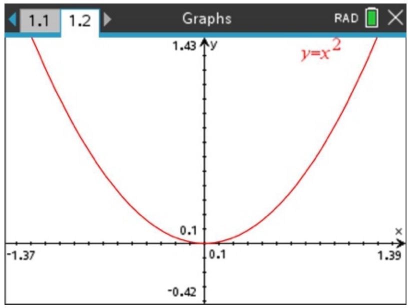
### Viewing and Interpreting Graphs

The points $(x, y)$ in the plane whose coordinates are the input-output pairs of a function $y = f(x)$ make up the function's **graph.** The graph of the function $y = x + 2$, for example, is the set of points with coordinates $(x, y)$ for which $y$ equals $x + 2$.

### Functions Defined by Equations

Equations in two variables can be used to define functions. However, not every equation in two variables represents a function.




### The Vertical Line Test

If a graph represents a function, then each input determines one and only one output. Thus, no two points can have the same $x$-coordinate and different $y$-coordinates. Since any two such points would lie on the same vertical line, this fact provides a useful test for determining whether a graph represents a function.

**A graph in a coordinate plane represents a function if and only if no vertical line intersects the graph more than once.**

# Reële functies wat anders bekeken

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \underbrace{f(x)}_{\text{Uitdrukking}} \end{aligned}$$

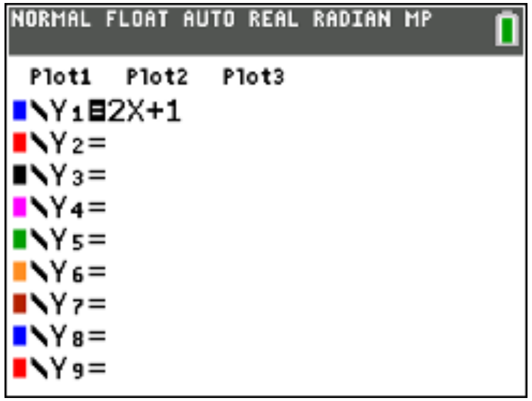
$$\underbrace{y = f(x)}_{\text{Vergelijking}}$$

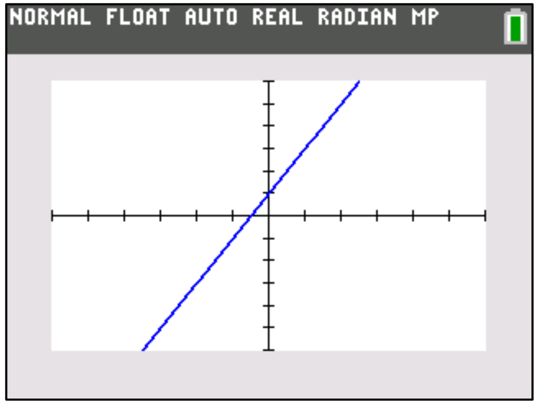
# Reëele functies wat anders bekeken

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \underbrace{2x+1}_{\text{Uitdrukking}} \end{aligned}$$

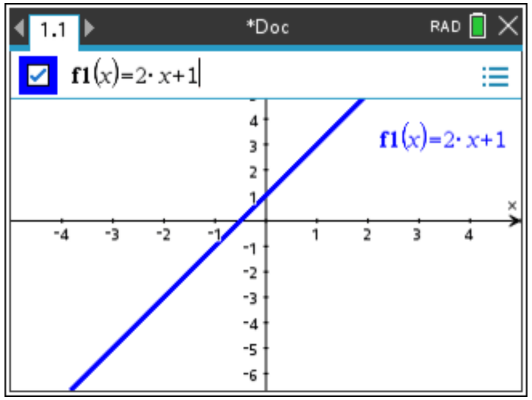
$$\underbrace{y = 2x + 1}_{\substack{\text{Vergelijking} \\ \text{vd grafiek}}}$$

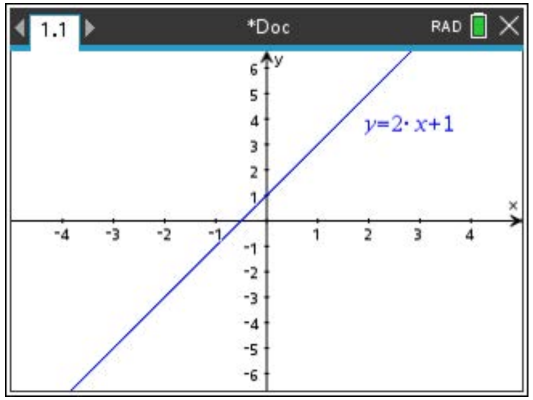
## Grafische Rekenmachine



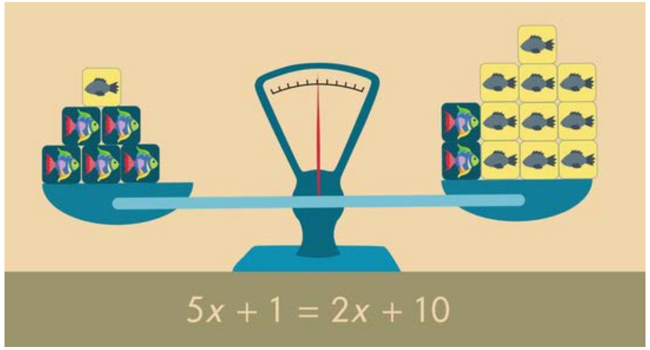

## Dynamische Wiskunde Software

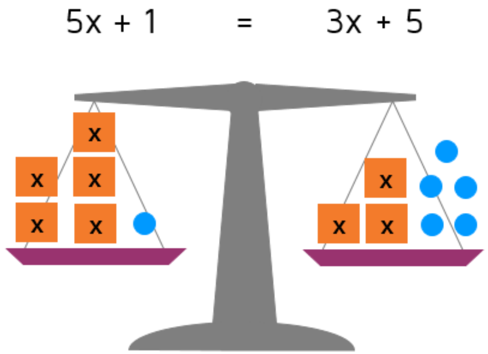


# Reëlle functies wat anders bekeken

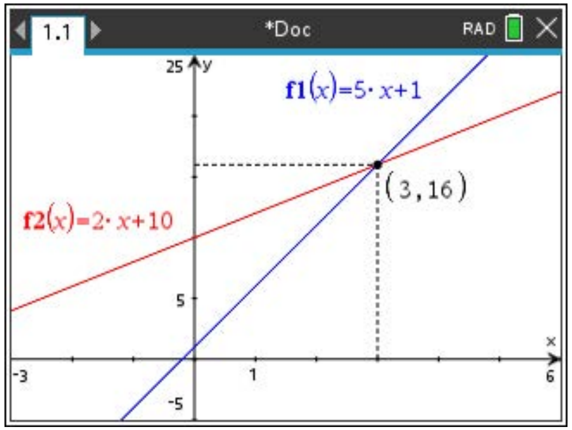
## *Vergelijkingen*

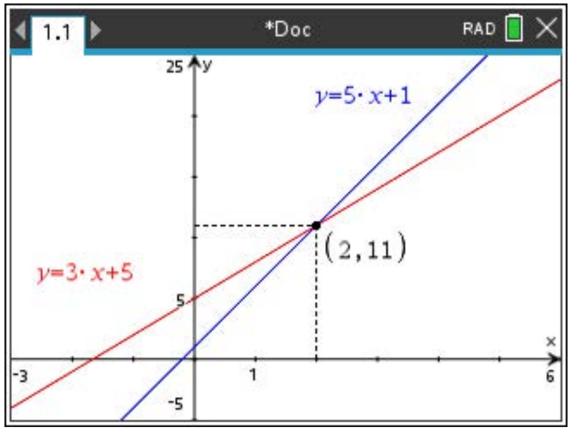



$$f_1 : x \longmapsto 5x + 1$$
$$f_2 : x \longmapsto 2x + 10$$

$$\begin{cases} y = 5x + 1 \\ y = 3x + 5 \end{cases} \qquad \begin{cases} 5x - y + 1 = 0 \\ 3x - y + 5 = 0 \end{cases}$$



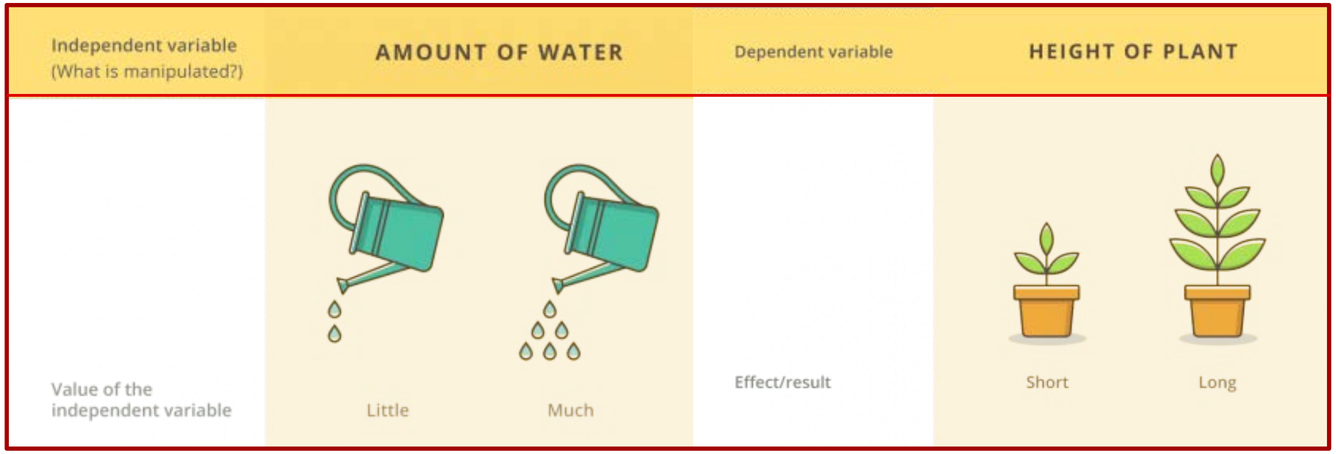



# Reëele functies wat anders bekeken

## *Variabelen*

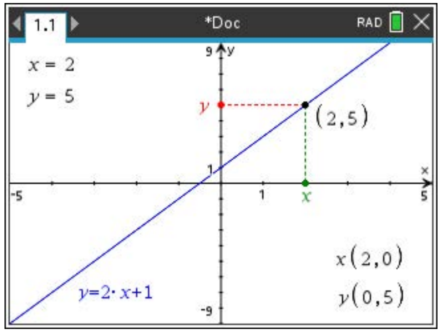
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \underbrace{2x+1}_{\text{Uitdrukking}}$$

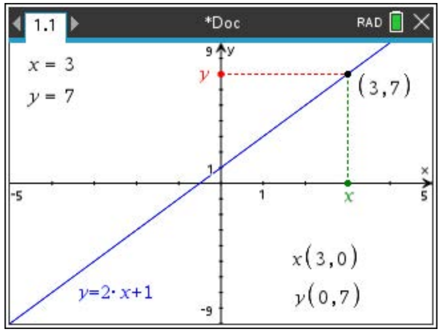
$$\underbrace{y = 2x + 1}_{\text{Vergelijking}}$$

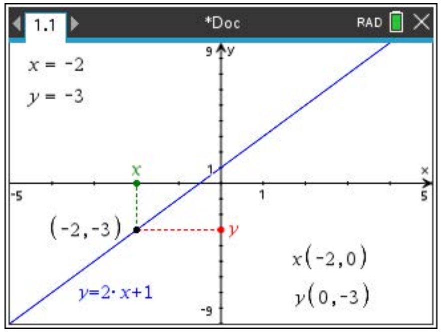


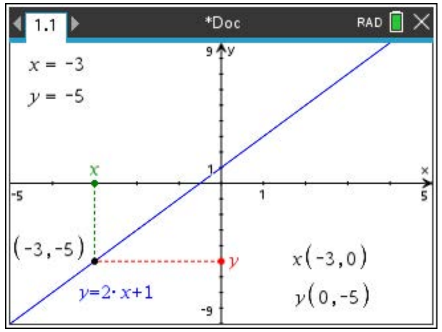

$x$-Variabele → Onafhankelijk

$y$-Variabele → Afhankelijk ⟶ $y(x) = 2x + 1$



# Reële functies wat anders bekeken
## *Variabelen*

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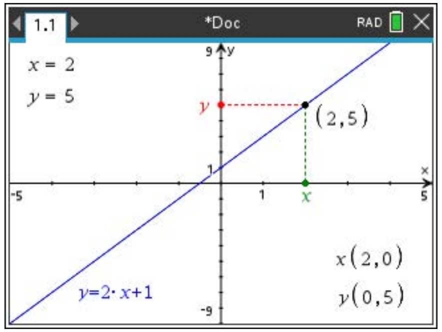
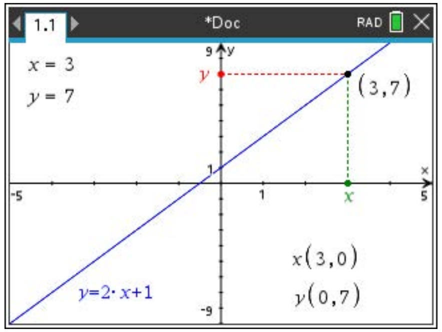
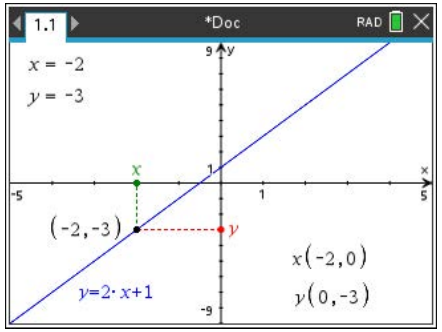


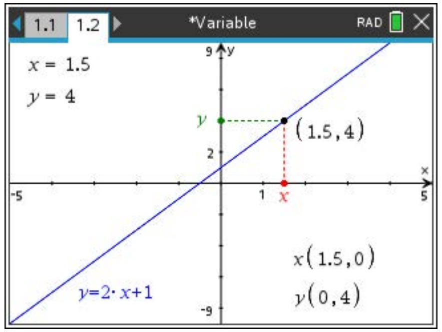



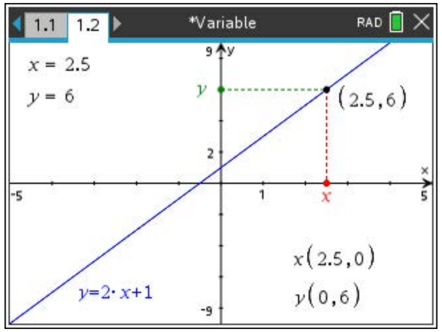



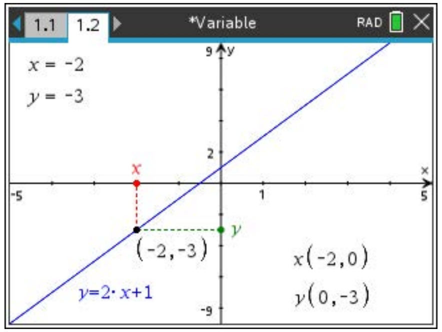

$x$-Variabele → Onafhankelijk

$y$-Variabele → Afhankelijk ⟶ $y(x) = 2x + 1$

Wat als we de rol van $x$ en $y$ omdraaien? $x(y)$?



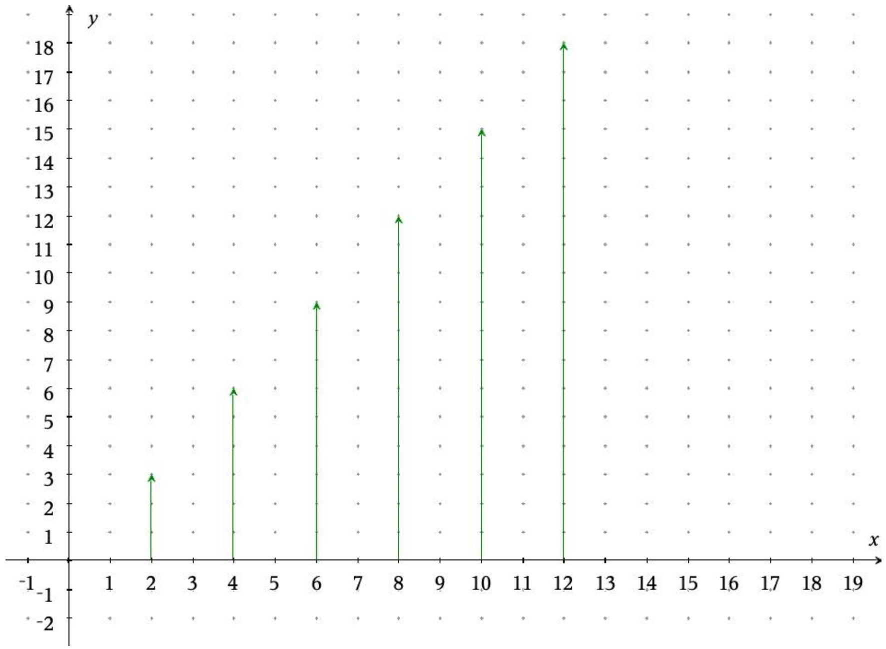



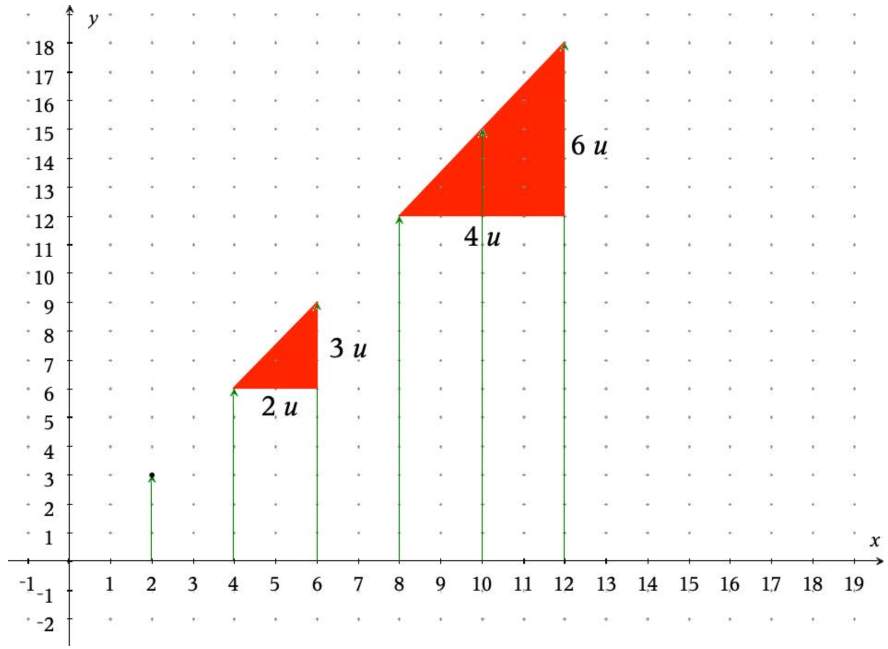

$$y = 2x + 1$$
$$\Updownarrow$$
$$x(y) = \frac{1}{2}y - \frac{1}{2}$$

# Reële functies wat anders bekeken
## *Helling*

$$\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}=\frac{12}{18}=\ldots$$

$(2,3) \blacktriangleright (4,6) \blacktriangleright (6,9) \blacktriangleright (8,12) \blacktriangleright (10,15) \blacktriangleright (12,18) \blacktriangleright \ldots$





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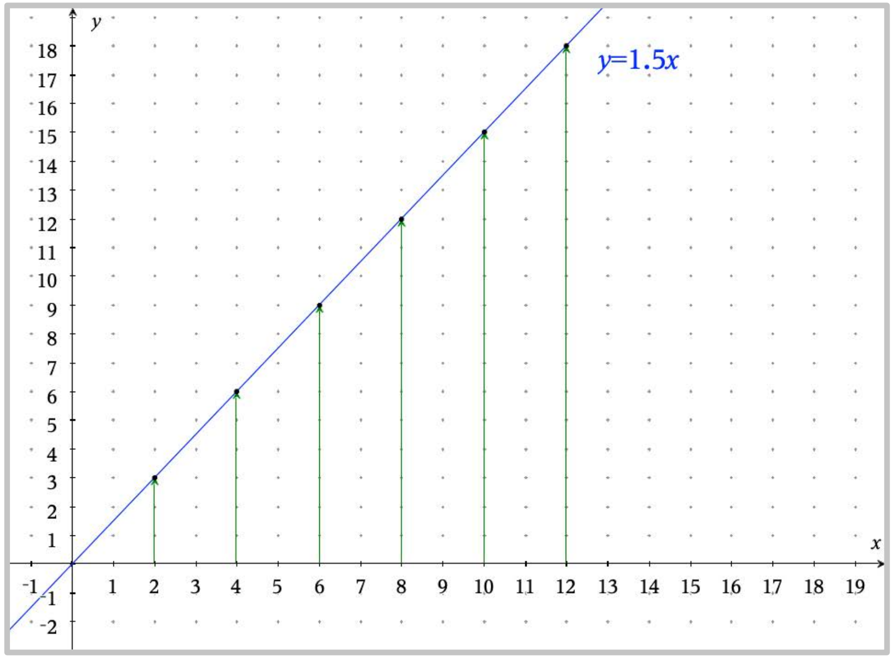
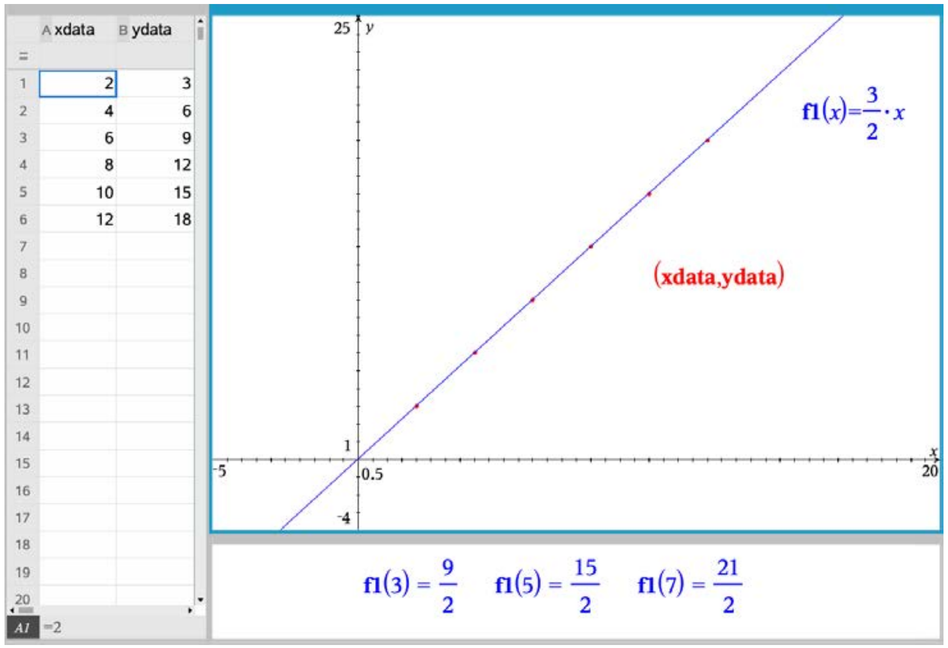
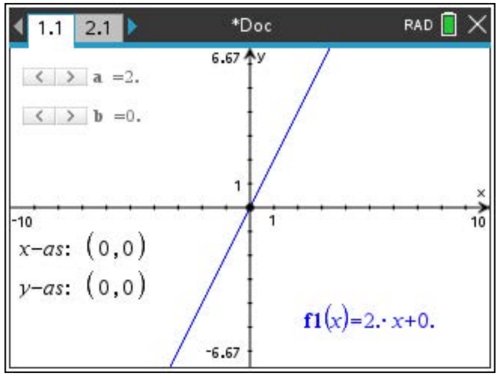


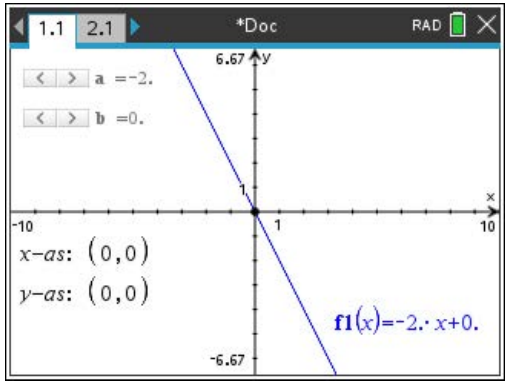



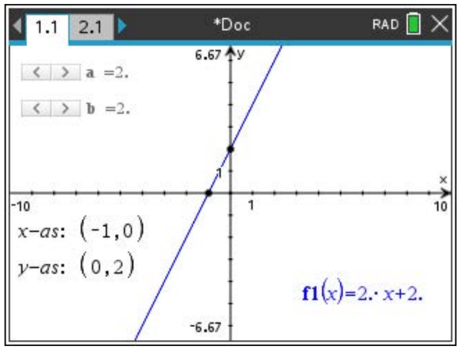
# Reële functies wat anders bekeken

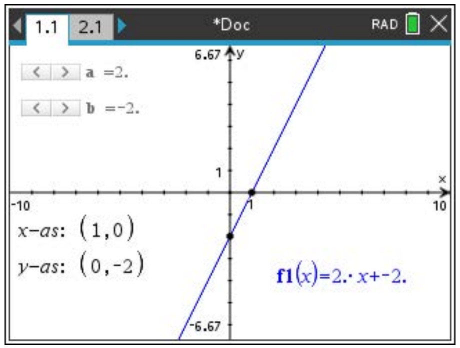
## *Functies van de 1e graad*

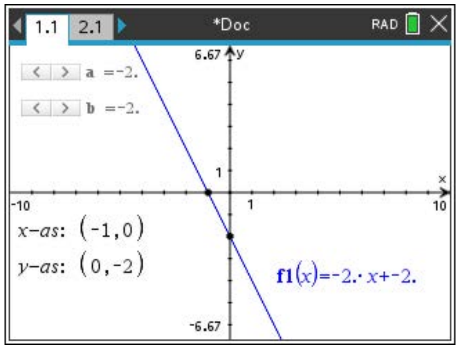
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto ax + b \text{ met } a \neq 0 \end{aligned}$$

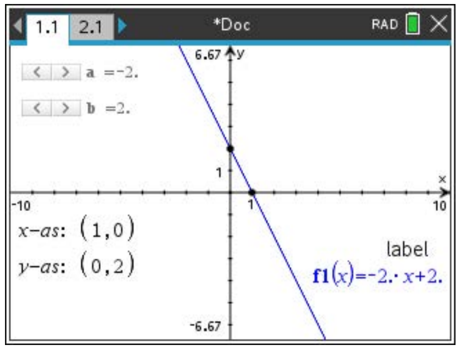


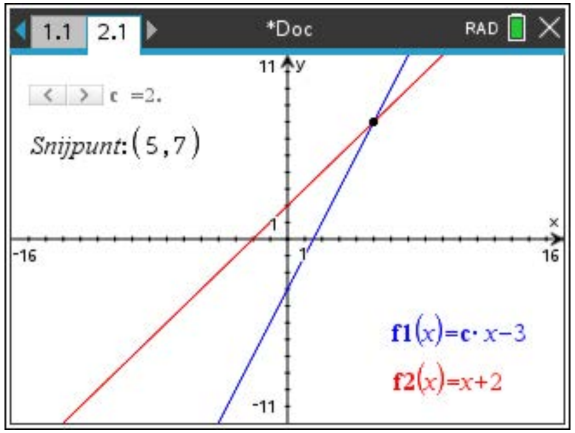



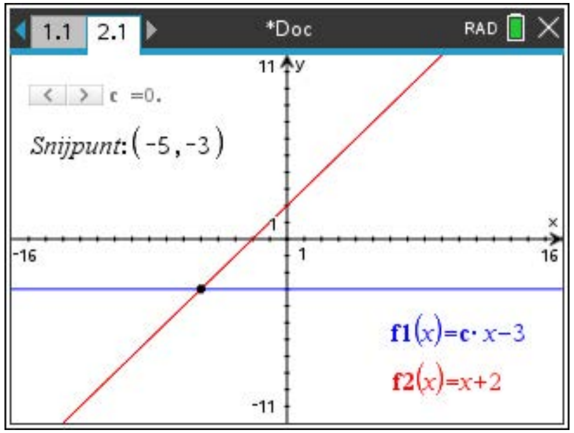
# Reële functies wat anders bekeken

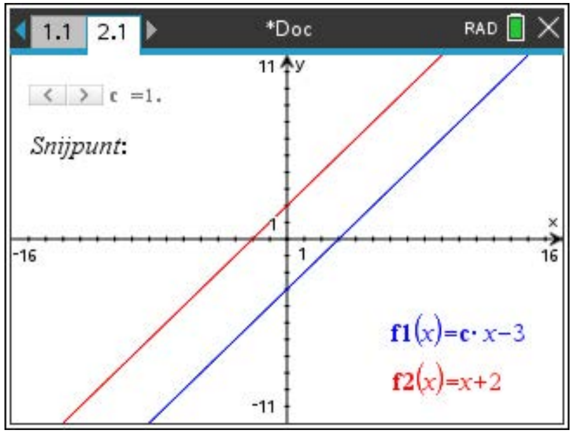
## *Vergelijkingen de 1e graad*

Bespreek de oplosbaarheid van $cx - 3 = x + 2$ i.f.v. de parameter $c \in \mathbb{R}$.

$f_1 : x \longmapsto cx + 3 \qquad f_2 : x \longmapsto x + 2$




$$cx - 3 = x + 2 \Leftrightarrow cx - x = 3 + 2 \Leftrightarrow (c-1)x - 5 = 0$$

$c = 1 \Rightarrow$ Geen oplossing

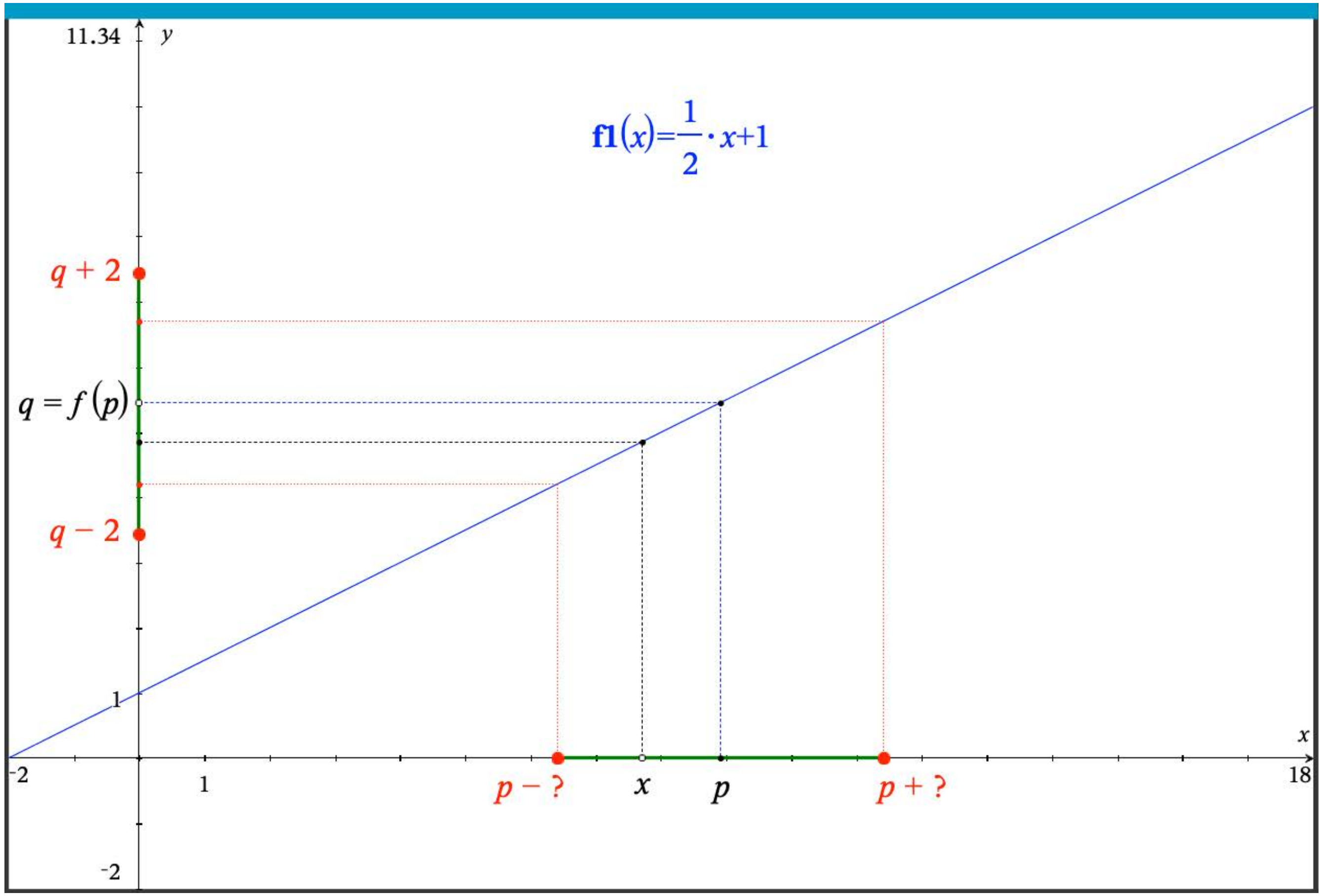
$c \neq 1 \Rightarrow$ Eén oplossing

$$x = \frac{5}{c-1}$$

# Reële functies wat anders bekeken
## *Afstand*

$$f : \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto \frac{1}{2}x + 1$$

Wat is de maximale afstand van $x$ tot $p$ zodat de afstand tussen $f(x)$ en $f(p)$ kleiner dan 2 blijft?



# Reële functies wat anders bekeken
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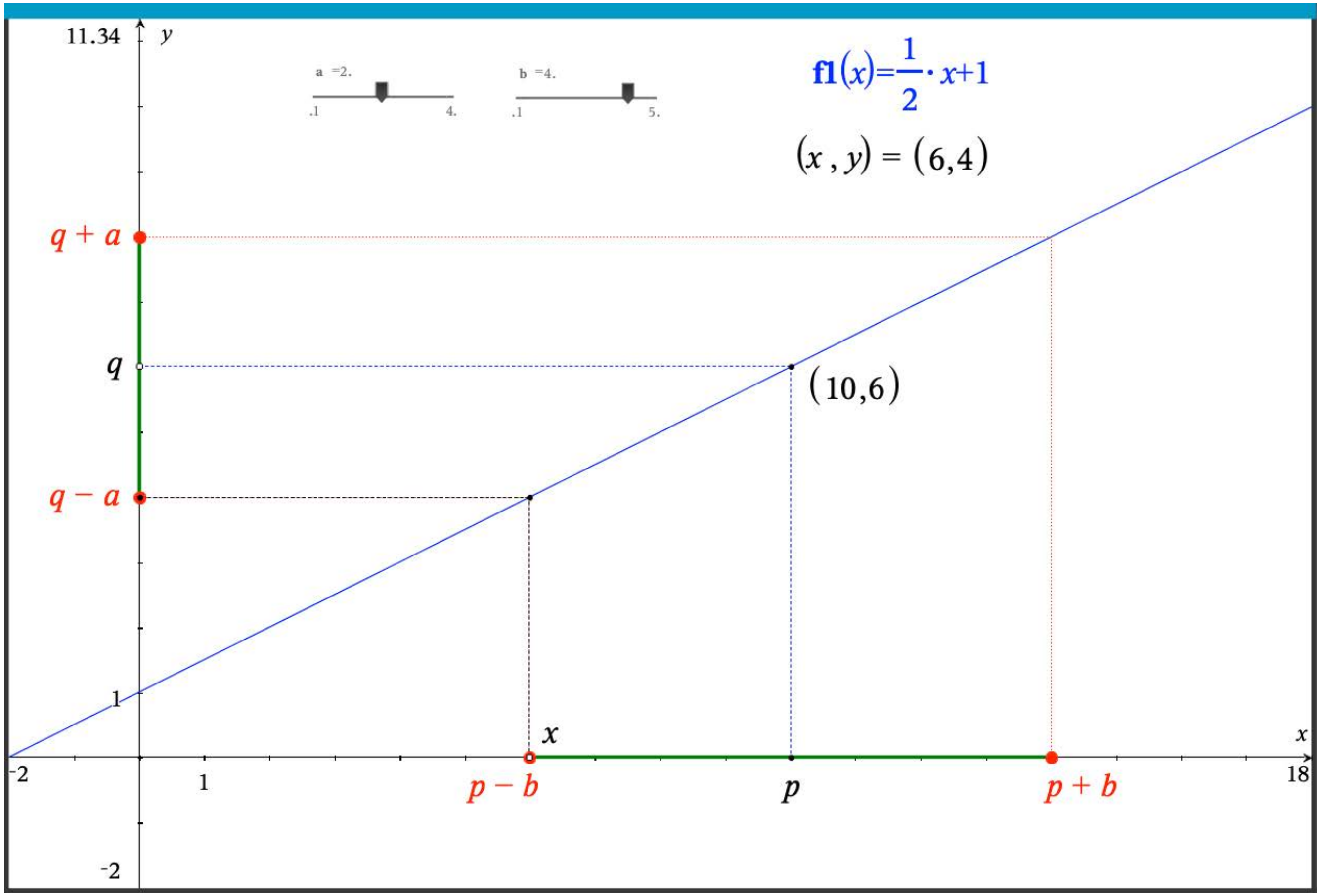
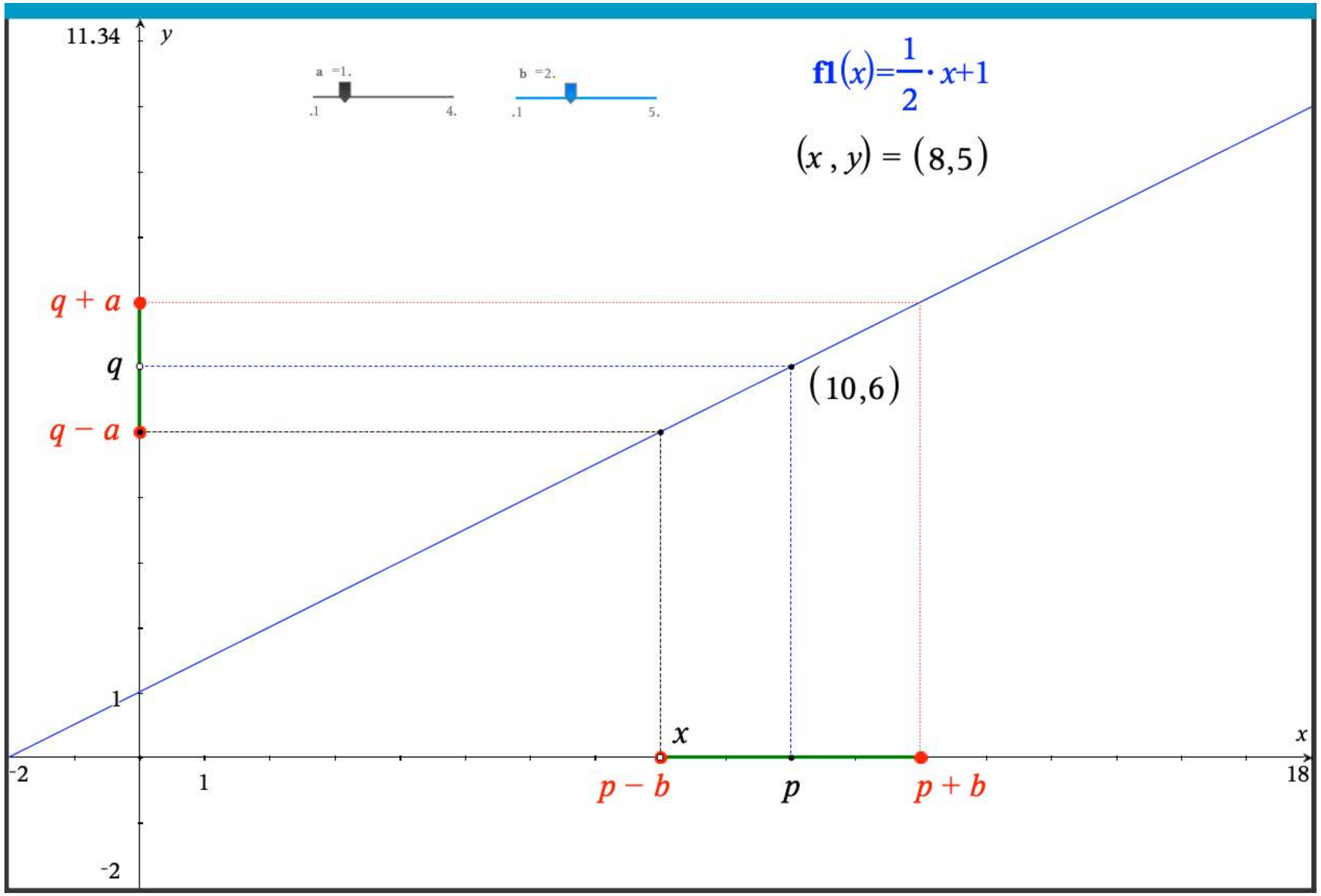


# Reële functies wat anders bekeken

## *Afstand*

$$f : \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto \frac{1}{2}x + 1$$

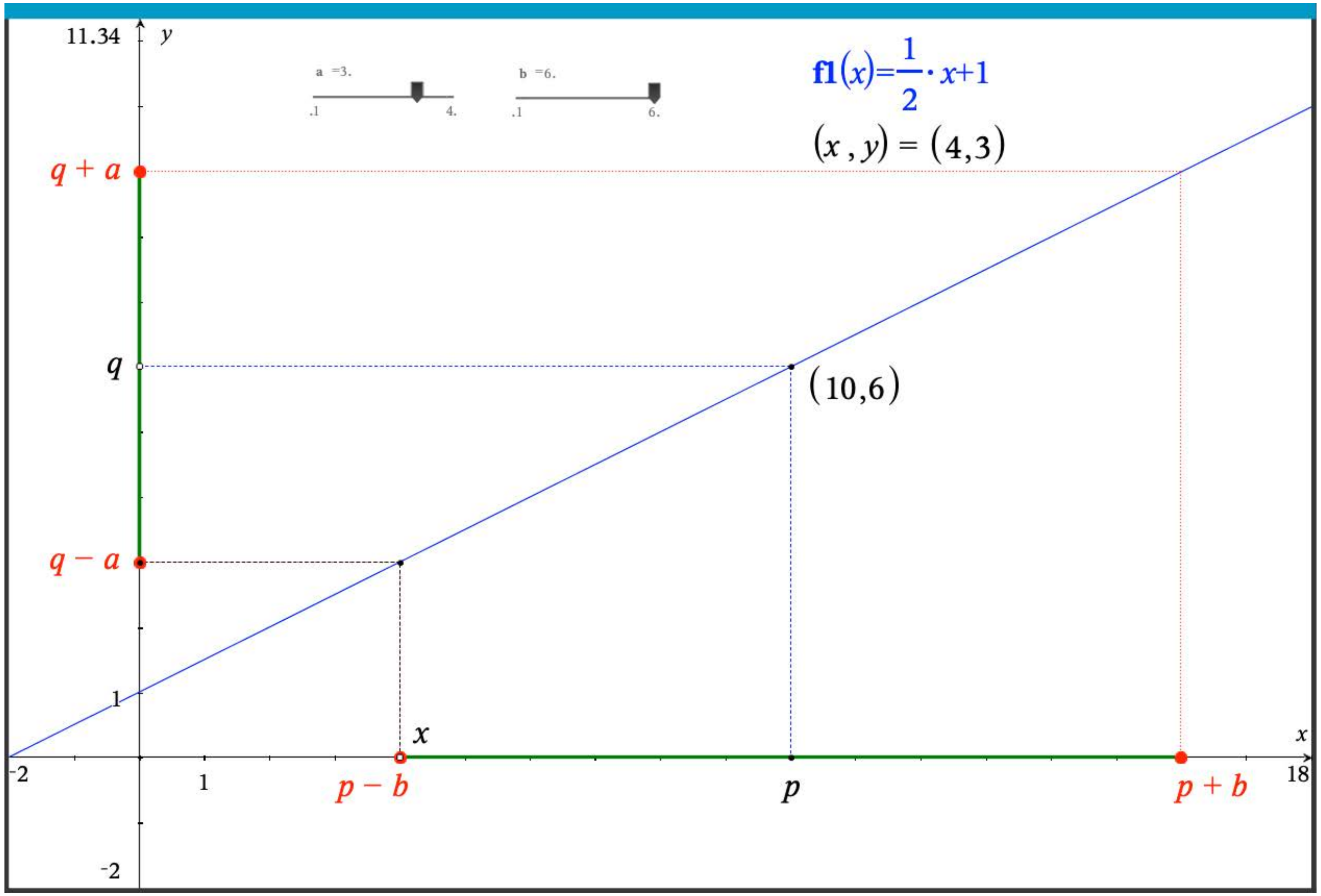
Wat is de maximale afstand van $x$ tot $p$ zodat de afstand tussen $f(x)$ en $f(p)$ kleiner dan 1 blijft ?



# Reële functies wat anders bekeken
## *Afstand*

$$f: \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto \frac{1}{2}x + 1$$

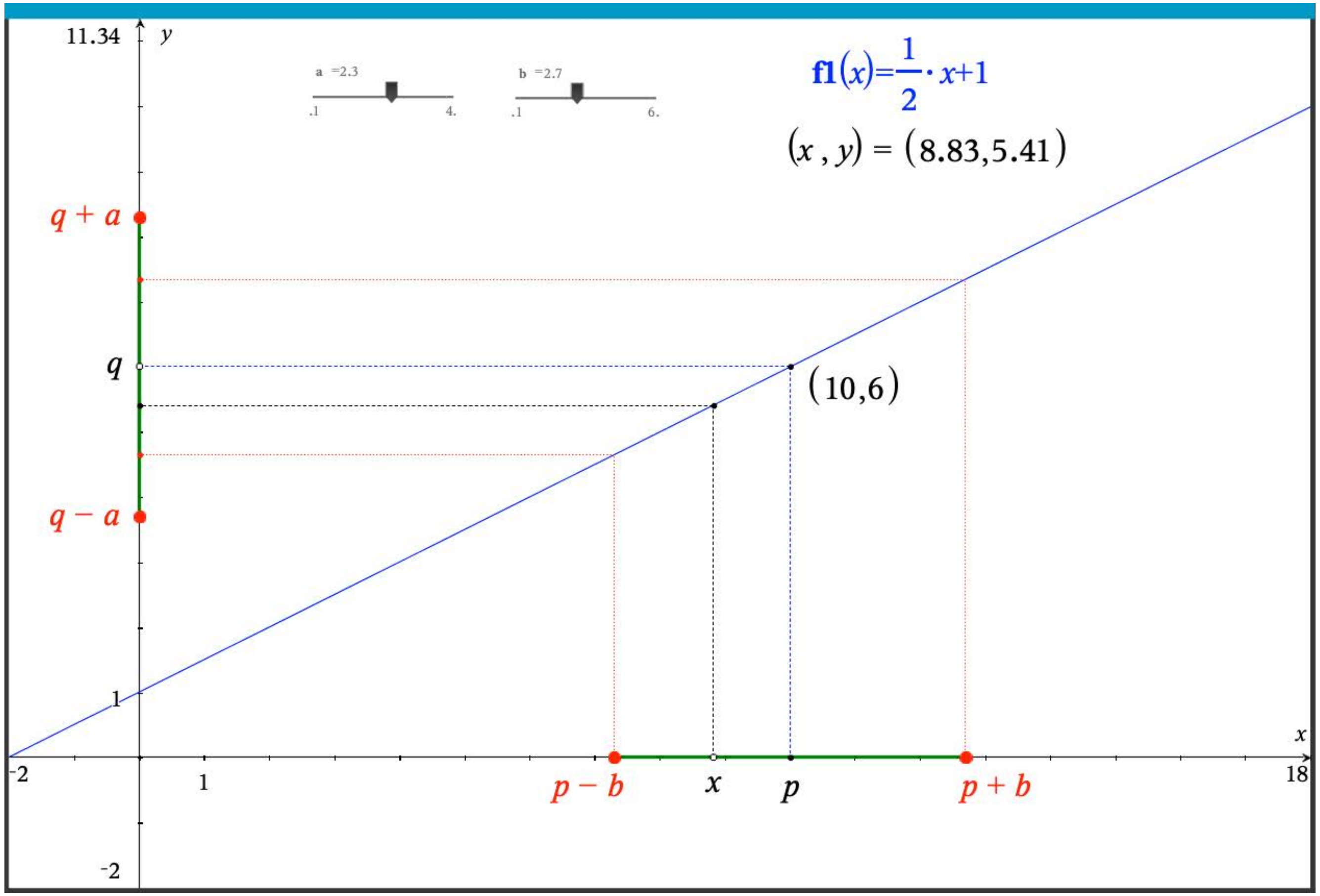
Wat is de maximale afstand van $x$ tot $p$ zodat
de afstand tussen $f(x)$ en $f(p)$ kleiner dan 3 blijft ?



# Reële functies wat anders bekeken
## *Afstand*

$$f : \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto \frac{1}{2}x + 1$$

Wat is de maximale afstand $b$ van $x$ tot $p$ zodat de afstand tussen $f(x)$ en $f(p)$ kleiner dan $a$ blijft ?




- $a = 1 \;\Rightarrow b = 2$
- $a = 2 \;\Rightarrow b = 4$
- $a = 3 \;\Rightarrow b = 6$

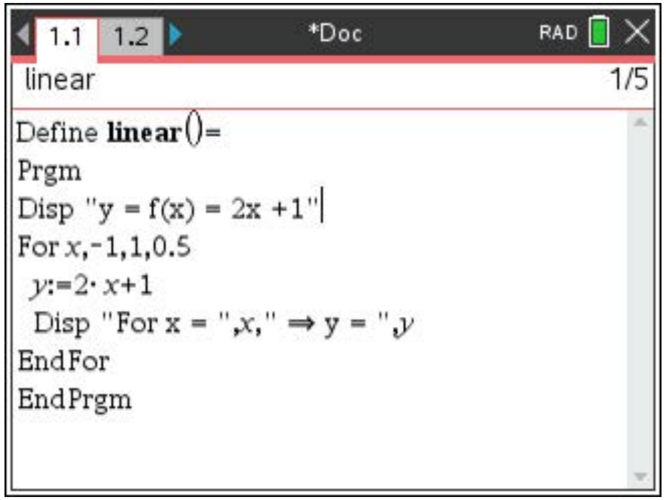
$$|f(x) - f(p)| < a$$
$$\Updownarrow$$
$$\frac{1}{2}|x - p| < a$$
$$\Updownarrow$$
$$|x - p| < 2a$$

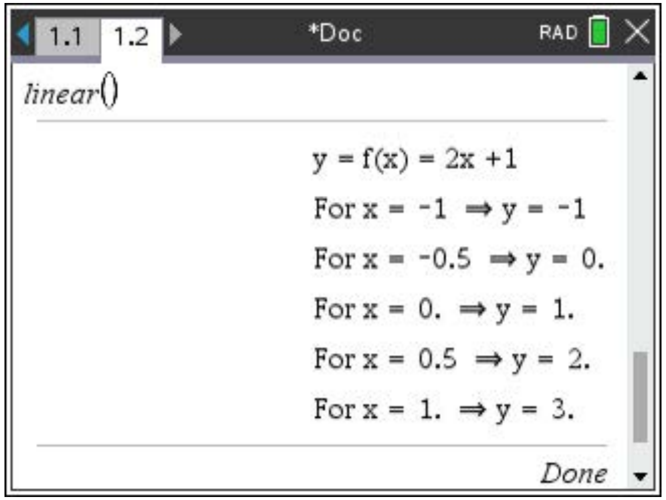
Voor iedere $a > 0$ de maximale afstand is $b = 2a$.

**A variable is like a storage box for various types of information**

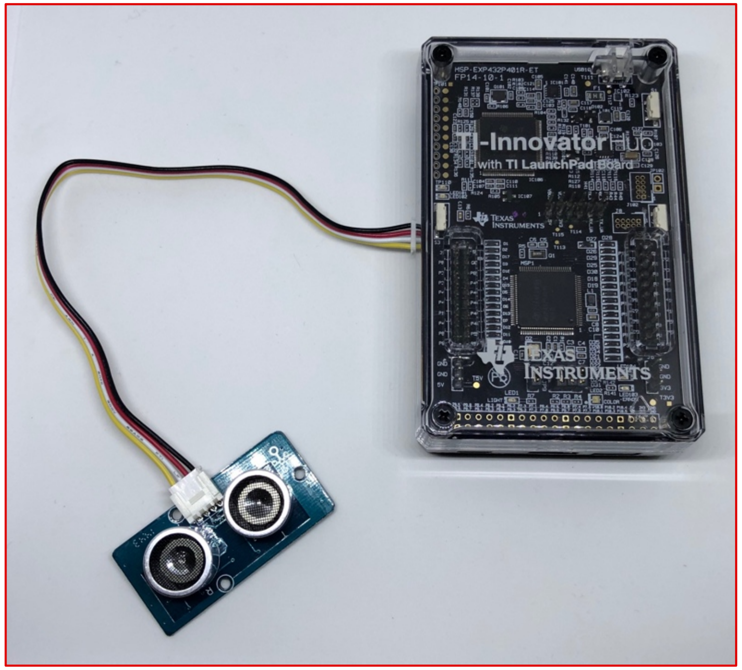
**Above all, you can always change the value of a variable!**

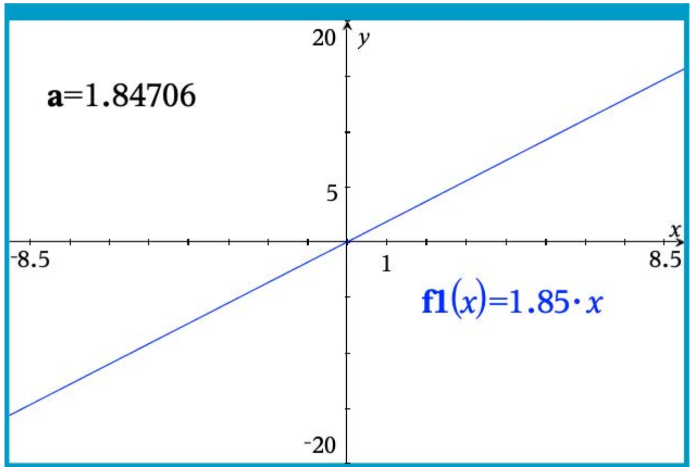
**A variable stores information.**



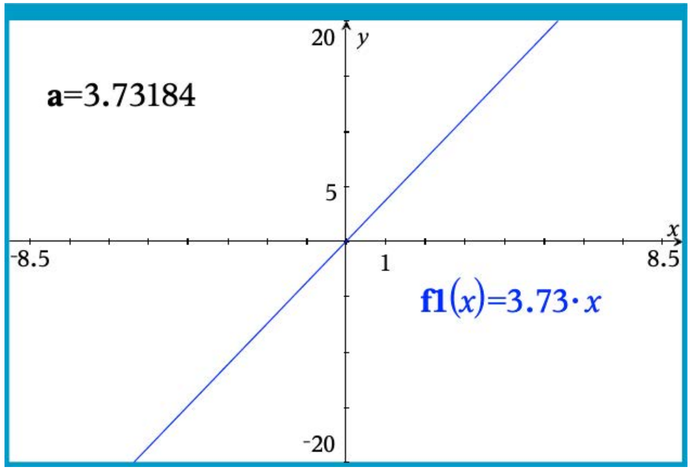
# Reële functies wat anders bekeken
## *Variabelen & Programmeren*



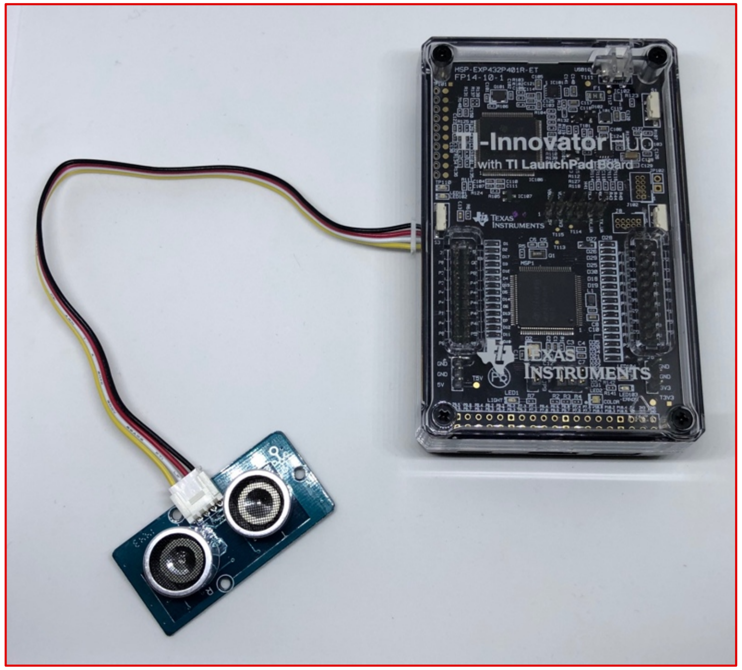

```
afstand                                   1/6
Define afstand()=
Prgm
Send "CONNECT RANGER 1 TO  IN 1"
Send "READ RANGER 1"
Get d
Disp d," m"
a:=10·d
Disp a," dm"
EndPrgm
```

```
afstand()
                                   0.18  m
                                   1.85  dm
                                      Done
afstand()
                                   0.37  m
                                   3.73  dm
                                      Done
```



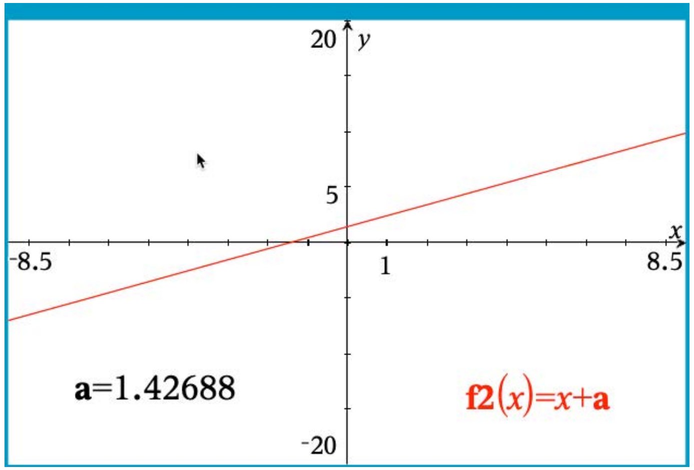
# Reële functies wat anders bekeken

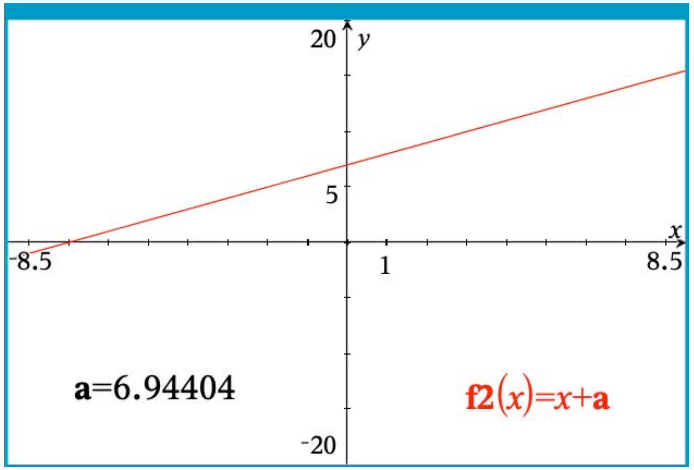
## *Variabelen & Programmeren*



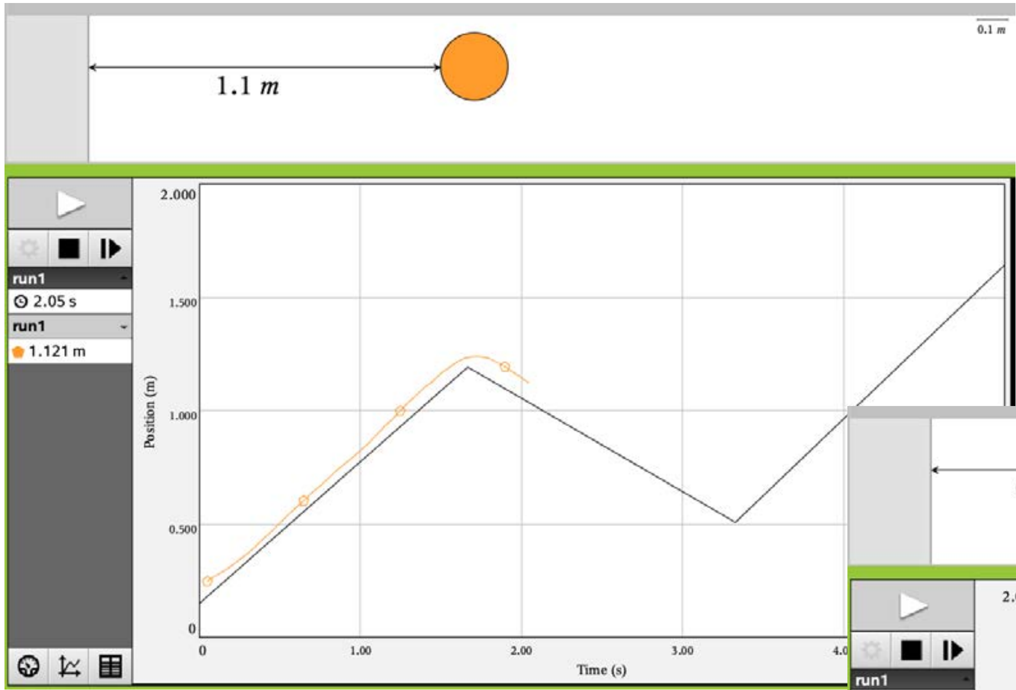

```
"afstand" stored successfully
Define afstand()=
Prgm
Send "CONNECT RANGER 1 TO  IN 1"
Send "READ RANGER 1"
Get d
Disp d," m"
a:=10·d
Disp a," dm"
EndPrgm
```

```
afstand()
                0.14  m
                1.43  dm
                    Done
afstand()
                0.69  m
                6.94  dm
                    Done
```

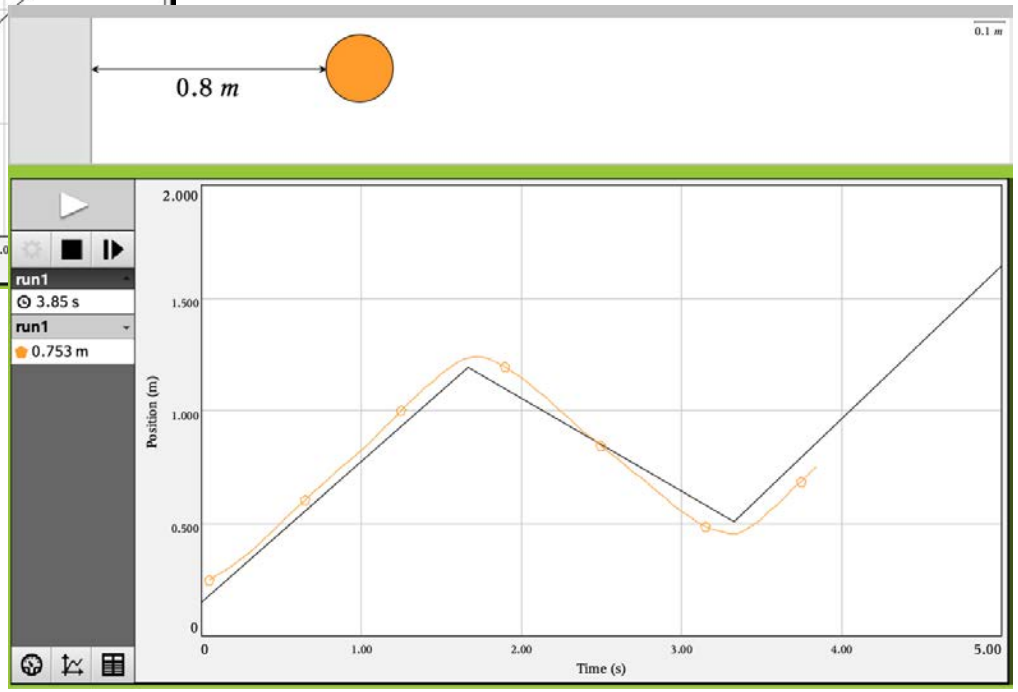


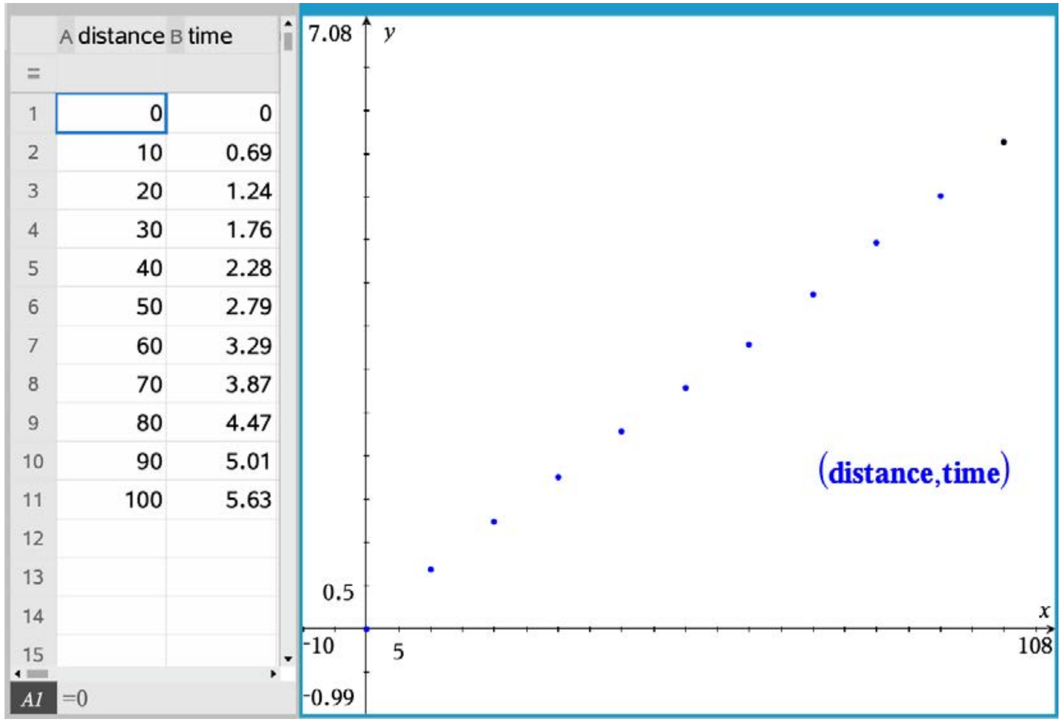
# Reële functies wat anders bekeken
## *Motion Match*





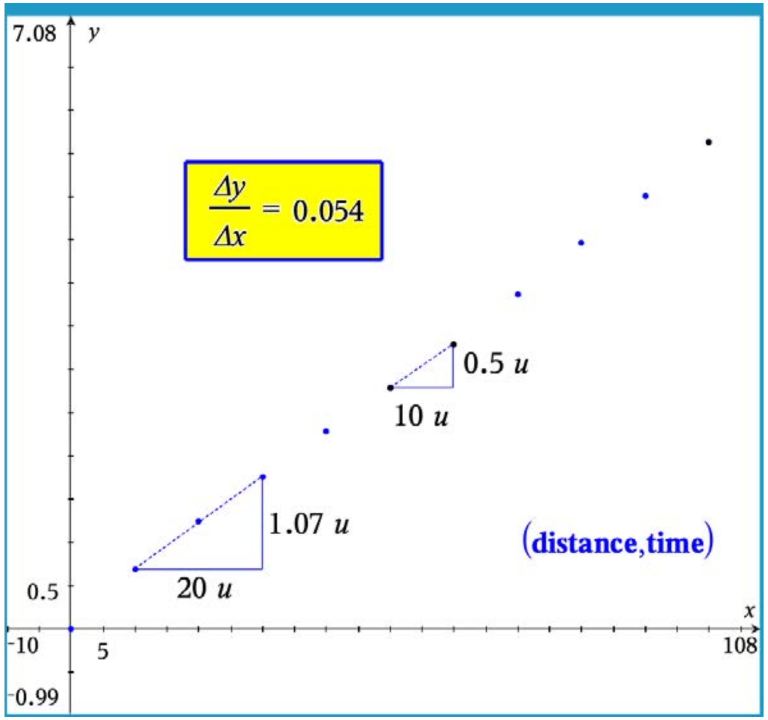



# Reële functies wat anders bekeken
## *Linear Rover*



| | A distance | B time |
|---|---|---|
| = | | |
| 1 | 0 | 0 |
| 2 | 10 | 0.69 |
| 3 | 20 | 1.24 |
| 4 | 30 | 1.76 |
| 5 | 40 | 2.28 |
| 6 | 50 | 2.79 |
| 7 | 60 | 3.29 |
| 8 | 70 | 3.87 |
| 9 | 80 | 4.47 |
| 10 | 90 | 5.01 |
| 11 | 100 | 5.63 |
| 12 | | |
| 13 | | |
| 14 | | |
| 15 | | |



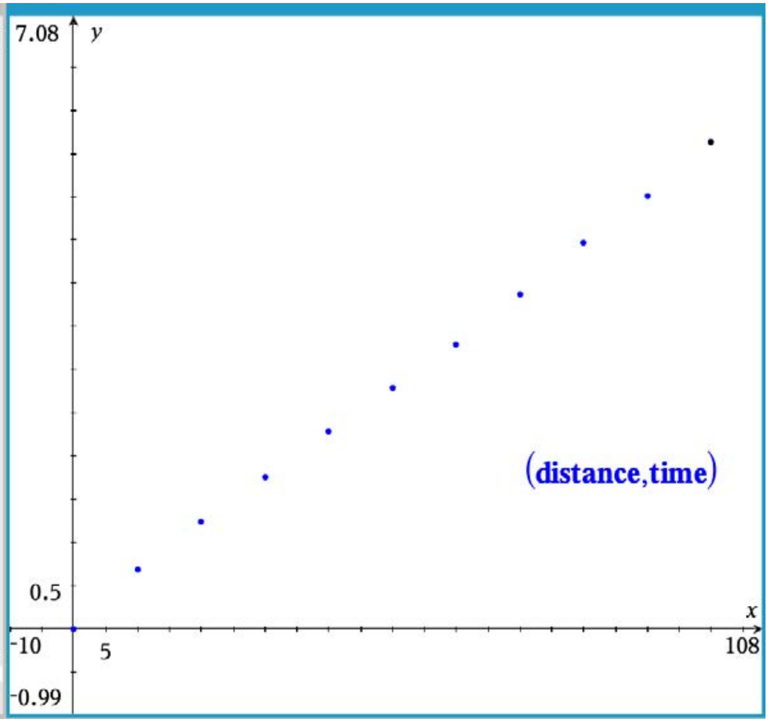


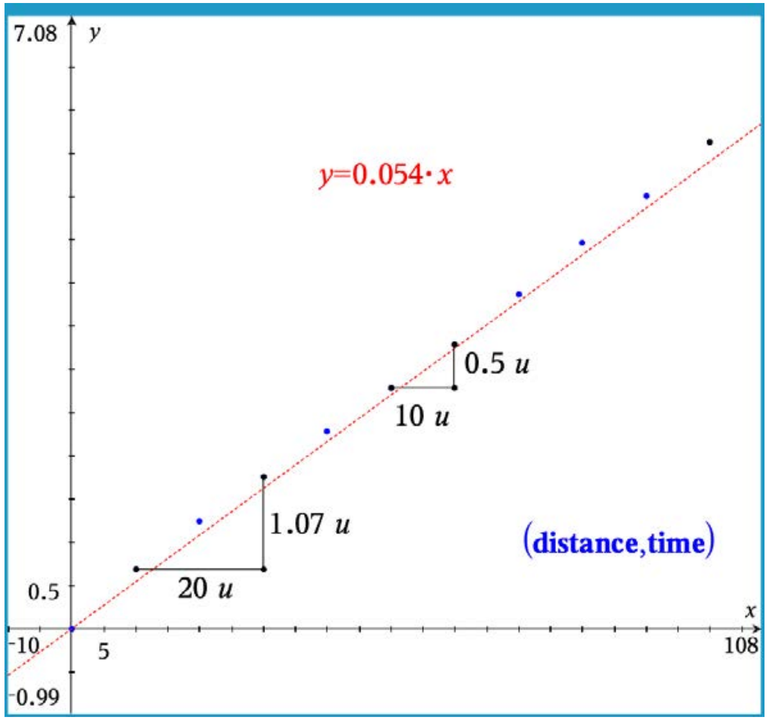

# Reeële functies wat anders bekeken

*Linear Rover*

| | A distance | B time |
|---|---|---|
| = | | |
| 1 | 0 | 0 |
| 2 | 10 | 0.69 |
| 3 | 20 | 1.24 |
| 4 | 30 | 1.76 |
| 5 | 40 | 2.28 |
| 6 | 50 | 2.79 |
| 7 | 60 | 3.29 |
| 8 | 70 | 3.87 |
| 9 | 80 | 4.47 |
| 10 | 90 | 5.01 |
| 11 | 100 | 5.63 |
| 12 | | |
| 13 | | |
| 14 | | |
| 15 | | |

A1 =0



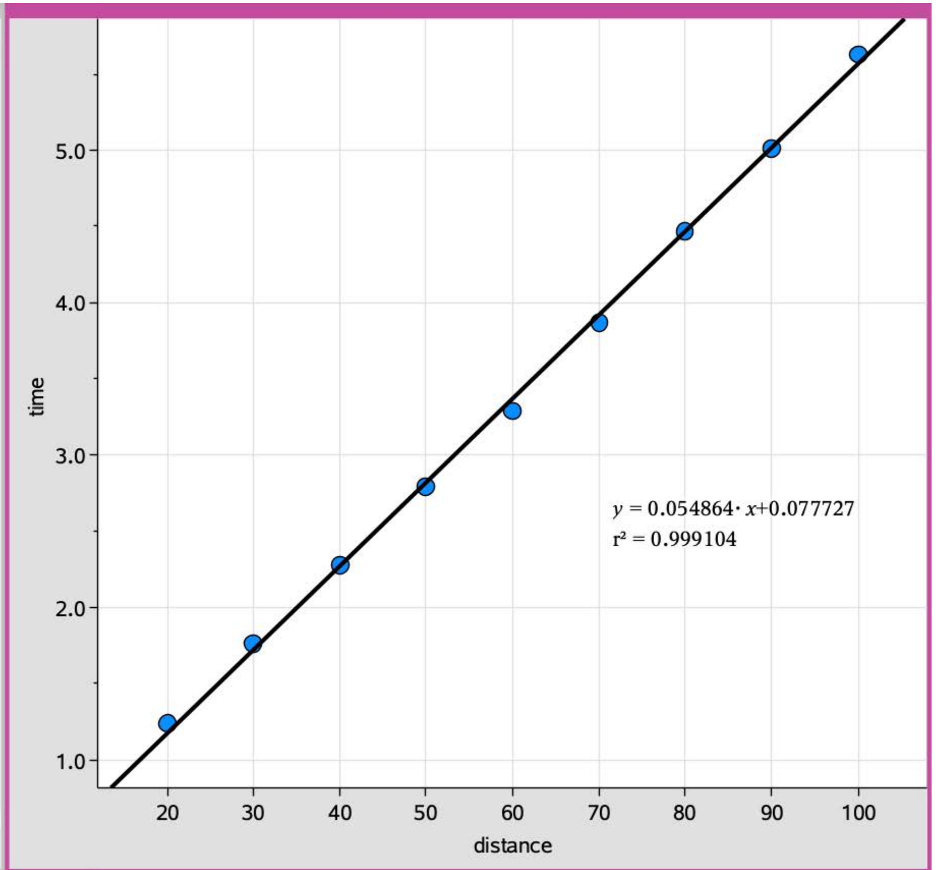




# Reële functies wat anders bekeken

## *Linear Rover*

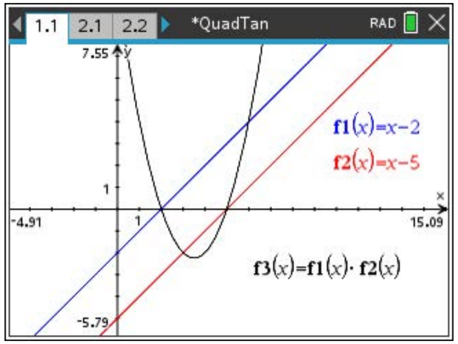
| | A distance | B time |
|---|---|---|
| = | distance | |
| 1 | 0 | 0 |
| 2 | 10 | 0.69 |
| 3 | 20 | 1.24 |
| 4 | 30 | 1.76 |
| 5 | 40 | 2.28 |
| 6 | 50 | 2.79 |
| 7 | 60 | 3.29 |
| 8 | 70 | 3.87 |
| 9 | 80 | 4.47 |
| 10 | 90 | 5.01 |
| 11 | 100 | 5.63 |
| 12 | | |
| 13 | | |
| 14 | | |
| 15 | | |

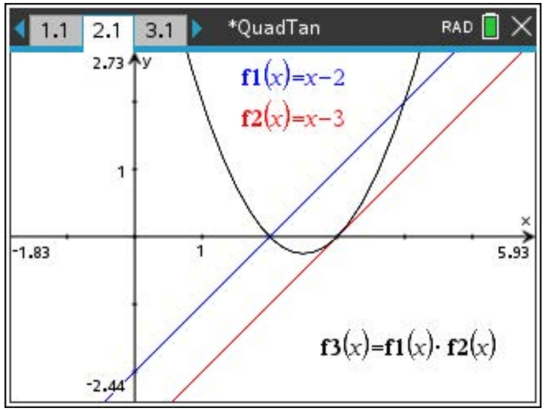
*A* distance

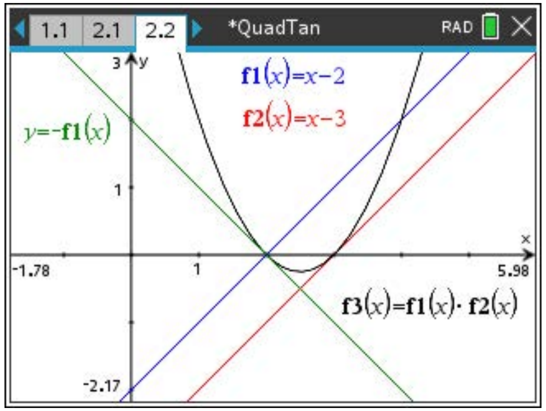


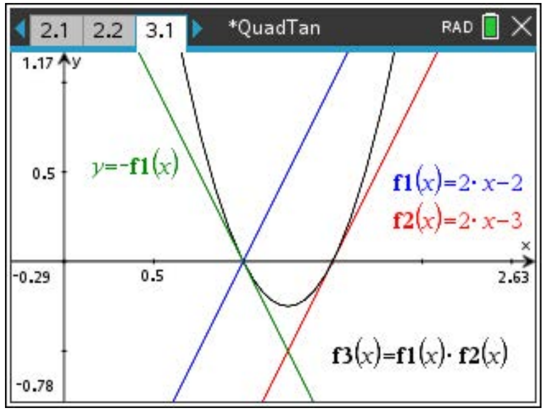

# Reële functies wat anders bekeken

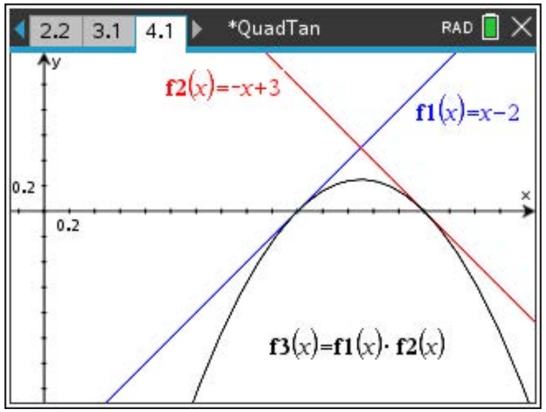
## *Van 1e graad naar 2e graad*

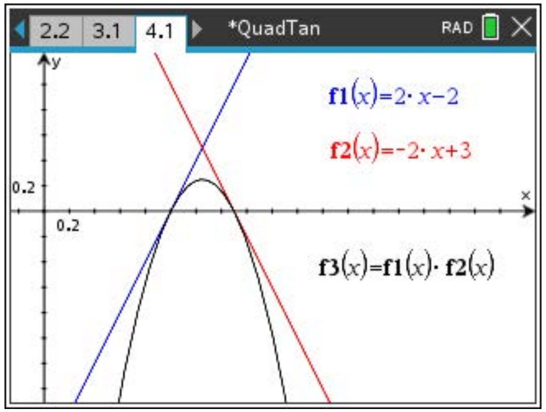


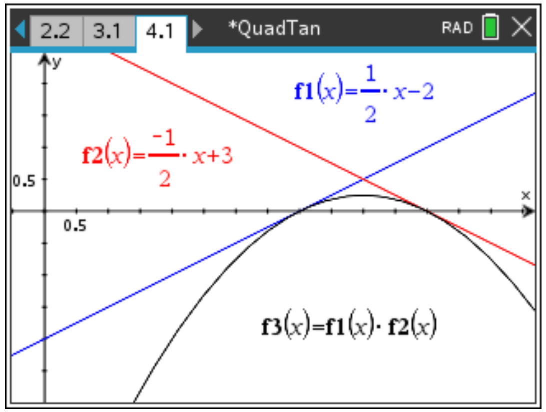



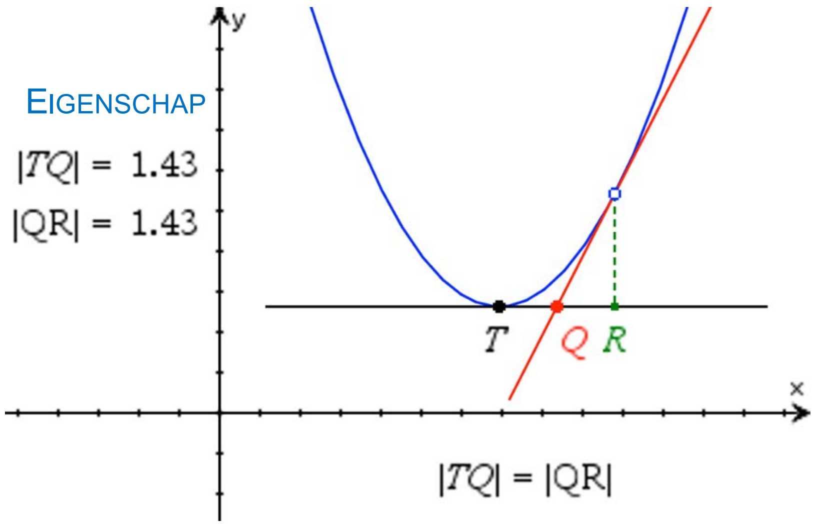



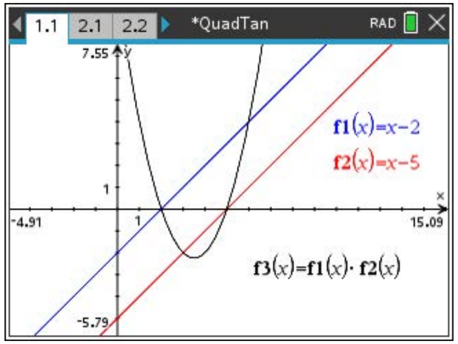



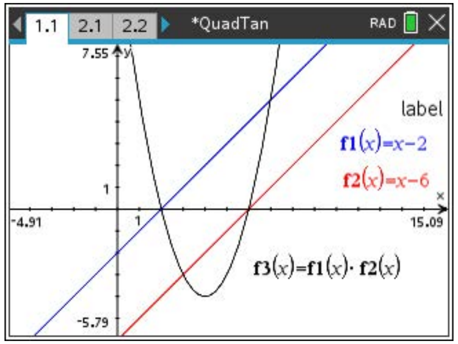
# Reële functies wat anders bekeken

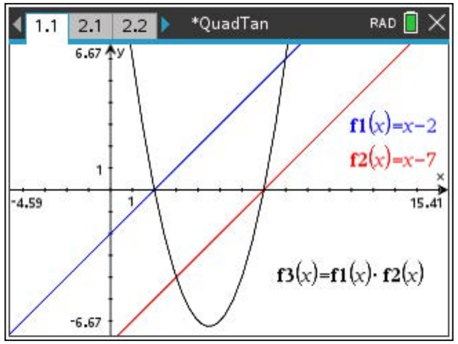
## *Van 1e graad naar 2e graad*

Voor welke $b \in \mathbb{R}$ geldt dat $y = (x - b)$ de raaklijn is aan de grafiek van $f(x) = (x-2)(x-b)$ in $(b, 0)$?



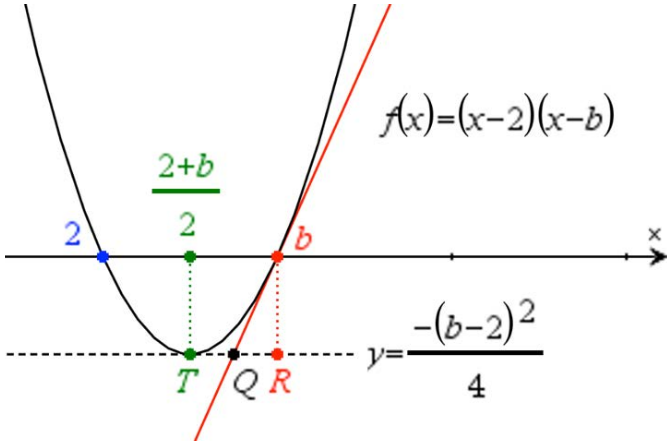



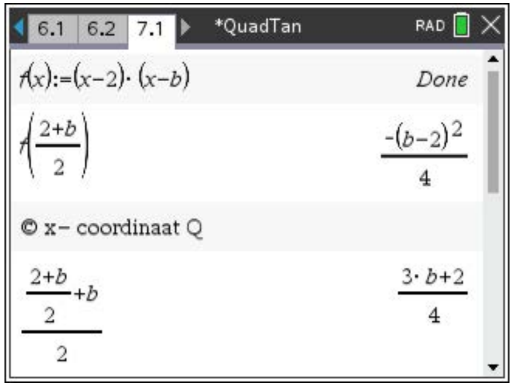
# Reeële functies wat anders bekeken

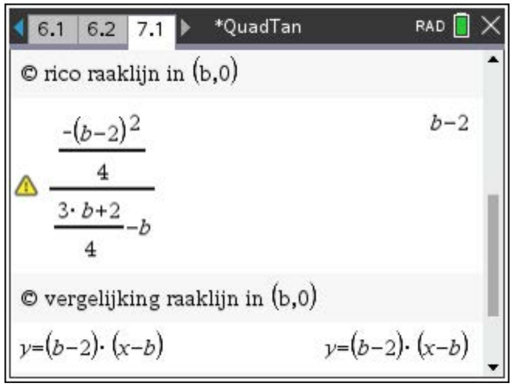
## *Van 1e graad naar 2e graad*

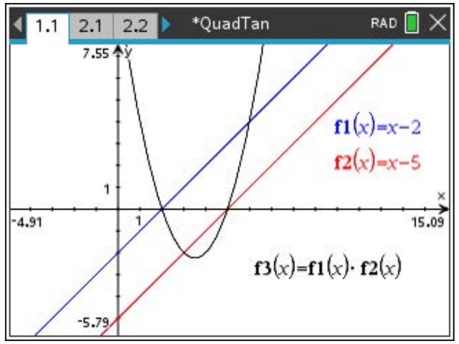
Voor welke $b \in \mathbb{R}$ geldt dat $y = (x-b)$ de raaklijn is aan de grafiek van $f(x) = (x-2)(x-b)$ in $(b, 0)$?

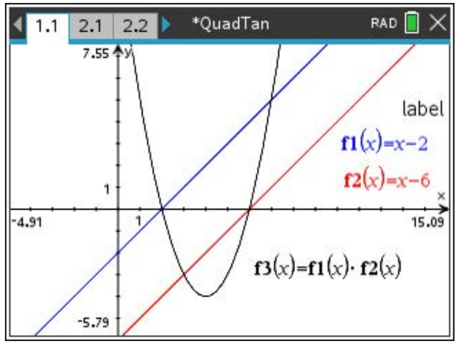


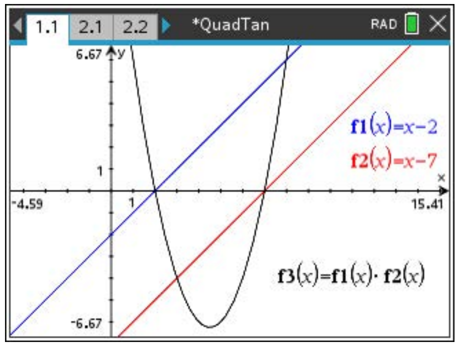

solve((b−2)·(x−b)=x−b,b) $\qquad$ b=x or b=3



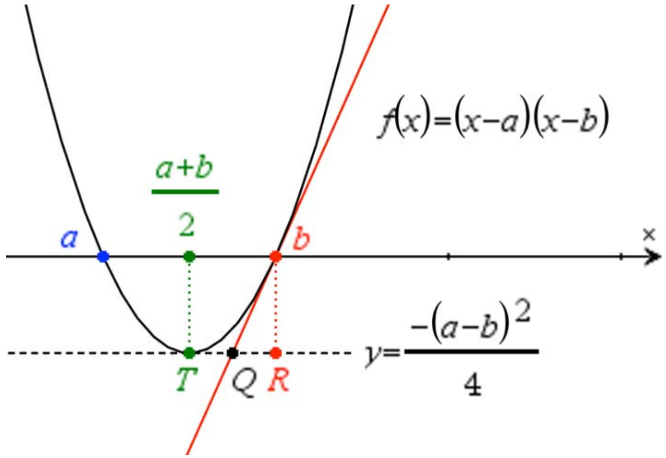



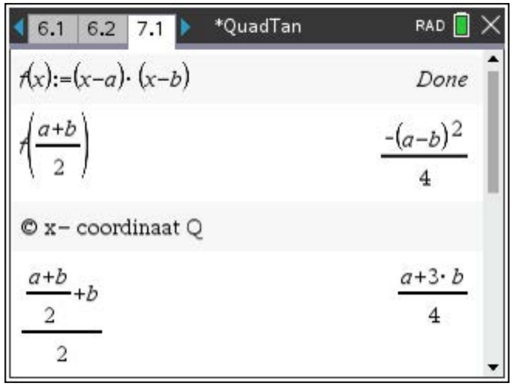
# Reële functies wat anders bekeken

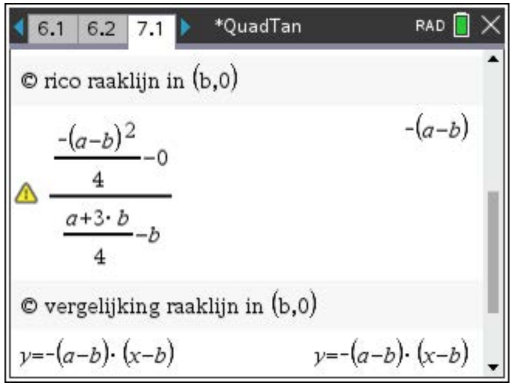
## *Van 1e graad naar 2e graad*

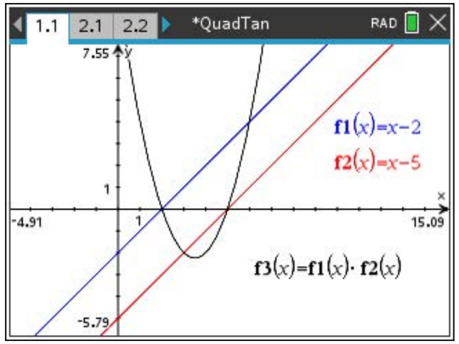
Voor welke a, $b \in \mathbb{R}$ geldt dat $y = (x - b)$ de raaklijn is aan de grafiek van $f(x) = (x - a)(x - b)$ in $(b, 0)$?

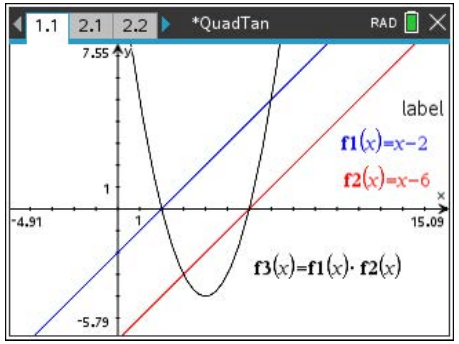


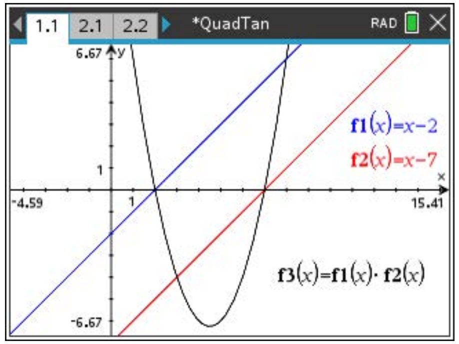

solve(-(a−b)·(x−b)=x−b,b) b=x or b=a+1



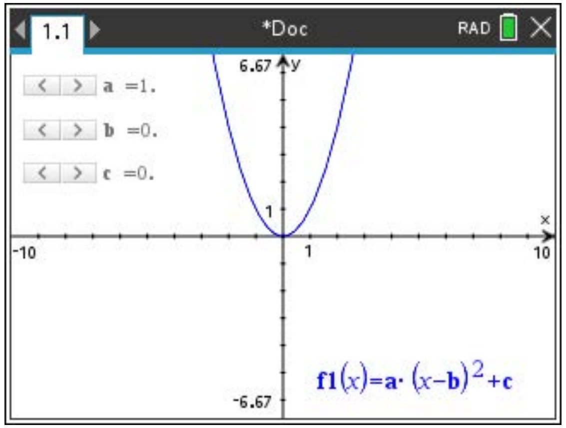



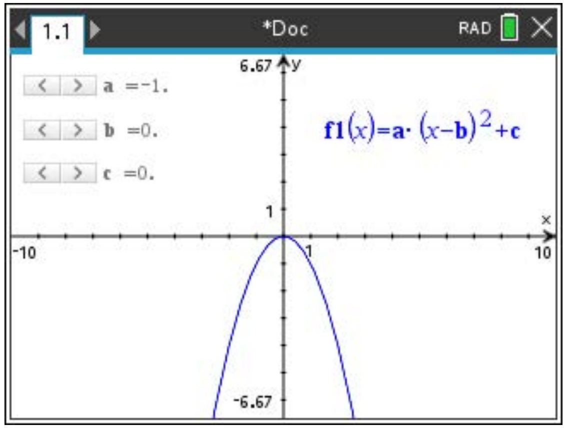
# Reële functies wat anders bekeken

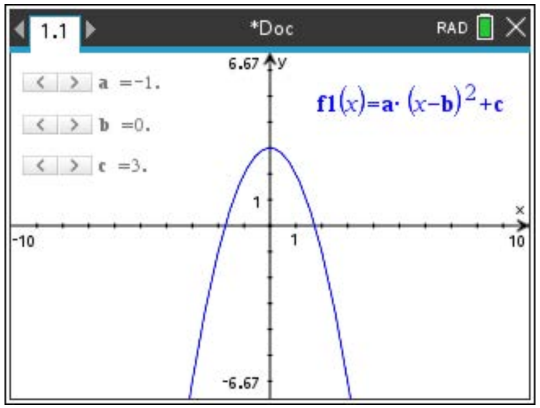
## *Functies van de 2e graad*

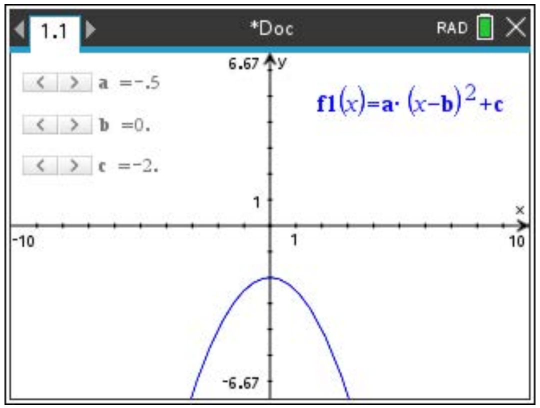
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto a(x-b)^2 + c \text{ met } a \neq 0$$

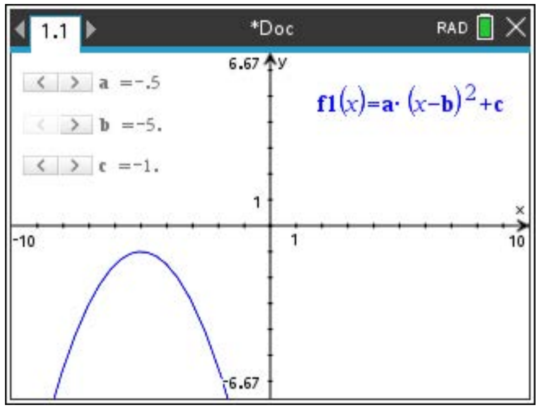
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto ax^2 + bx + c \text{ met } a \neq 0$$



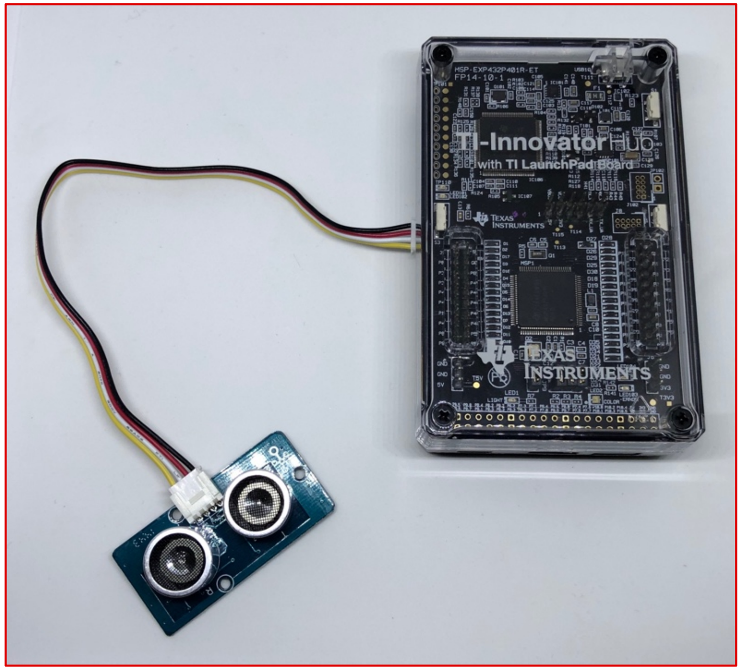



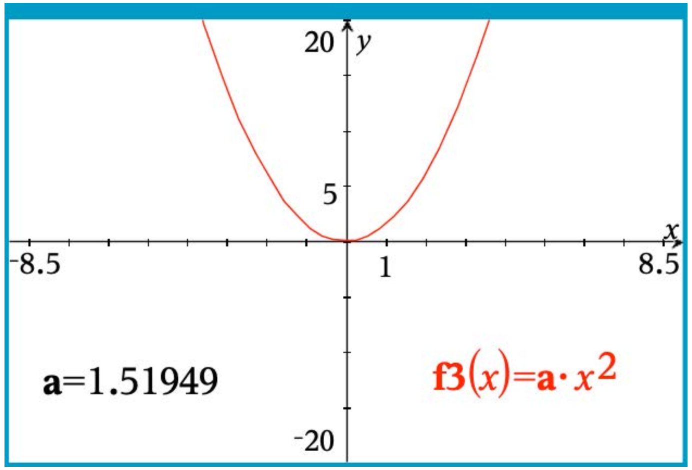



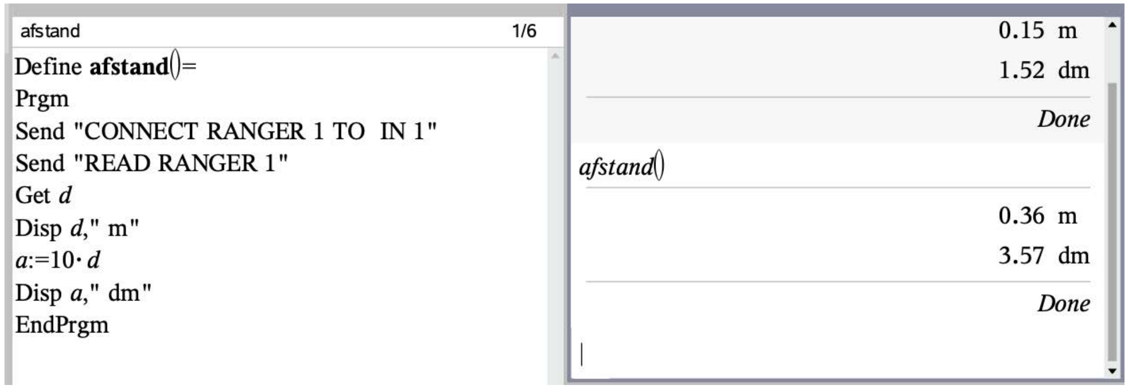



# Reële functies wat anders bekeken

## *Functies van de 2e graad*

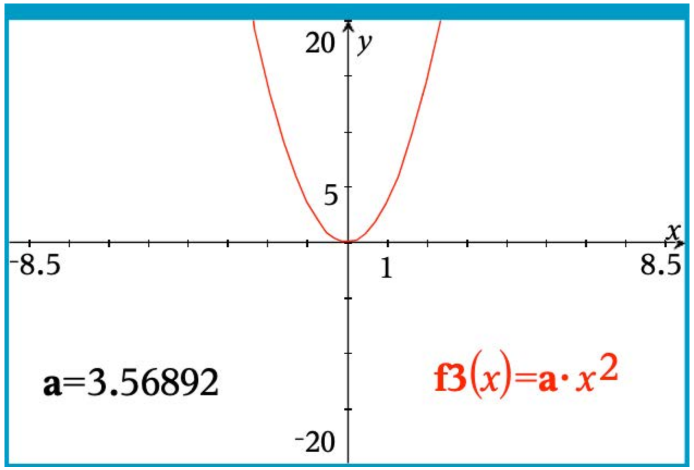


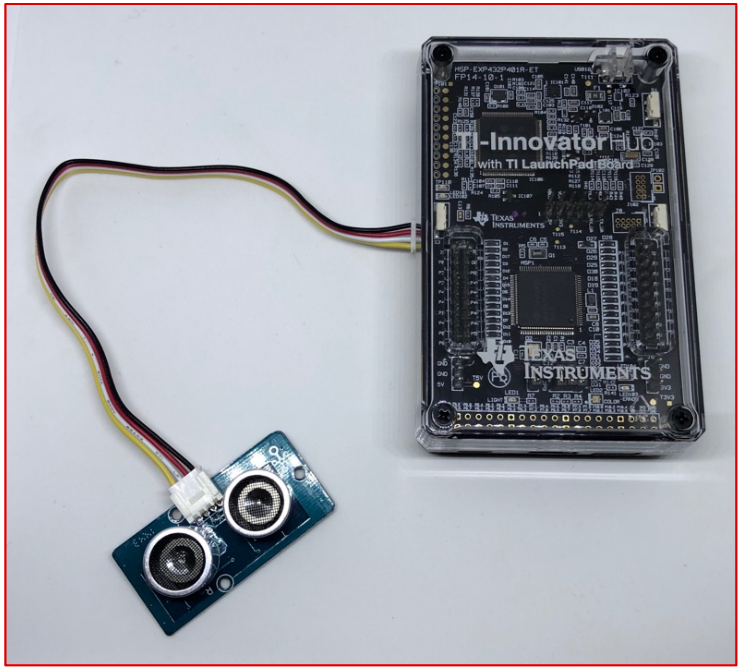
```
afstand                                  1/6
Define afstand()=
Prgm
Send "CONNECT RANGER 1 TO  IN 1"
Send "READ RANGER 1"
Get d
Disp d," m"
a:=10·d
Disp a," dm"
EndPrgm
```

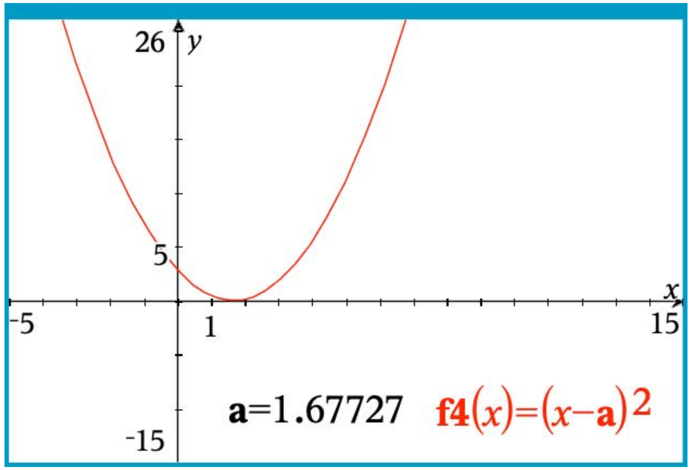
```
0.15  m
1.52  dm
Done
afstand()
0.36  m
3.57  dm
Done
```

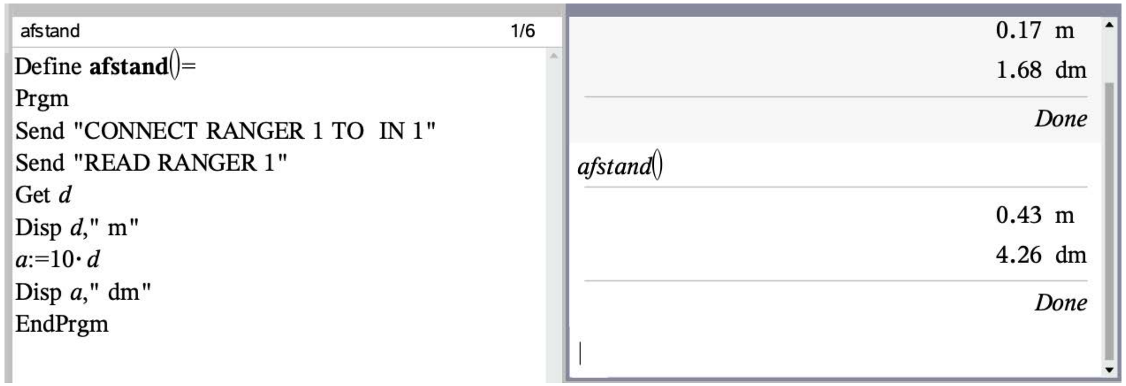


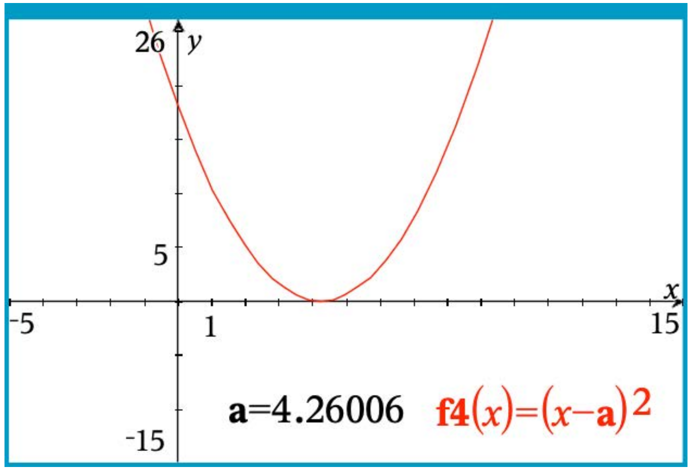
# Reële functies wat anders bekeken

## *Functies van de 2^e^ graad*



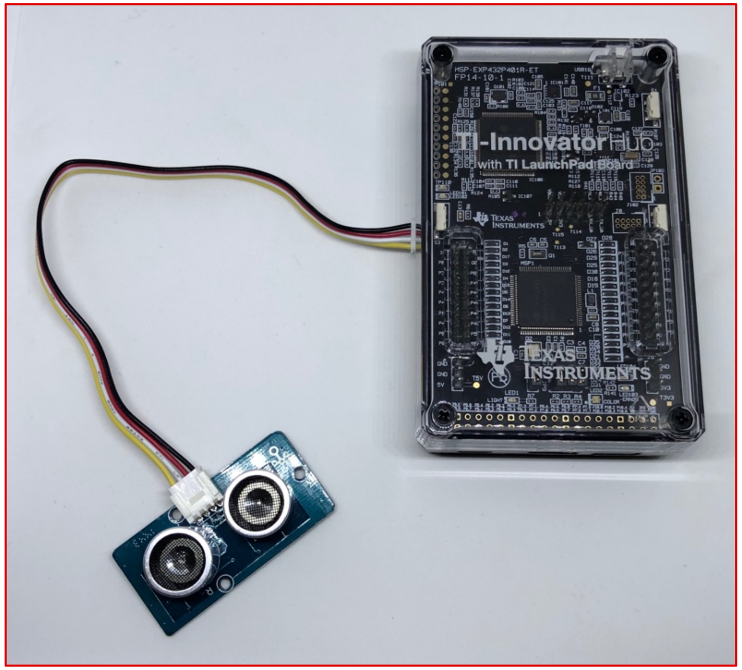



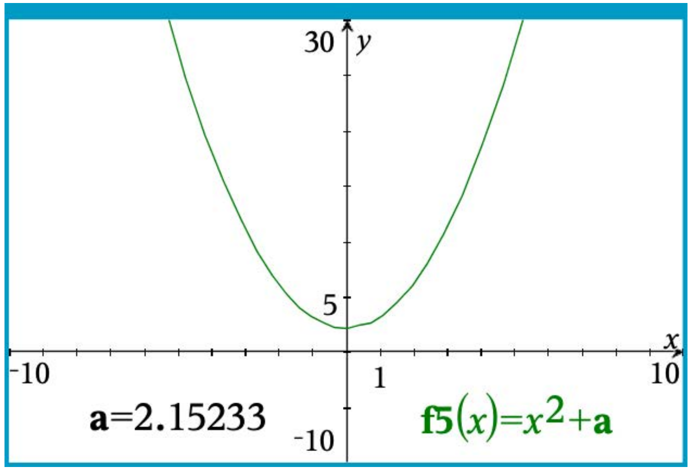



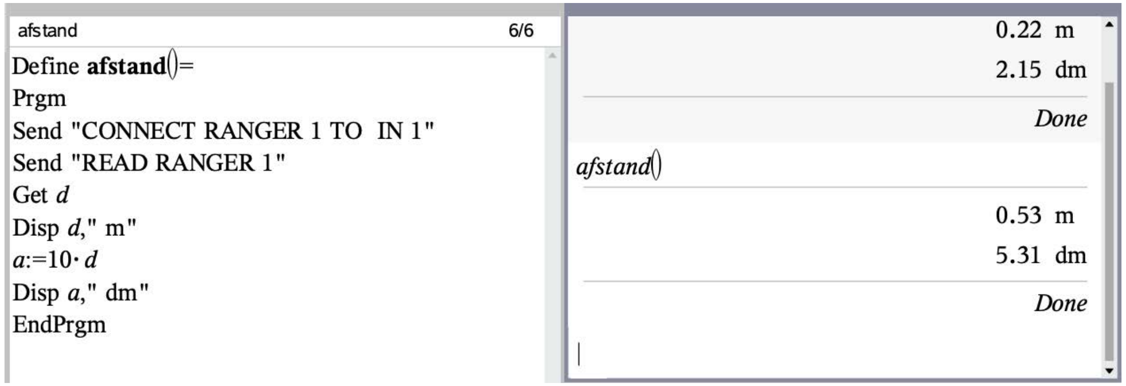



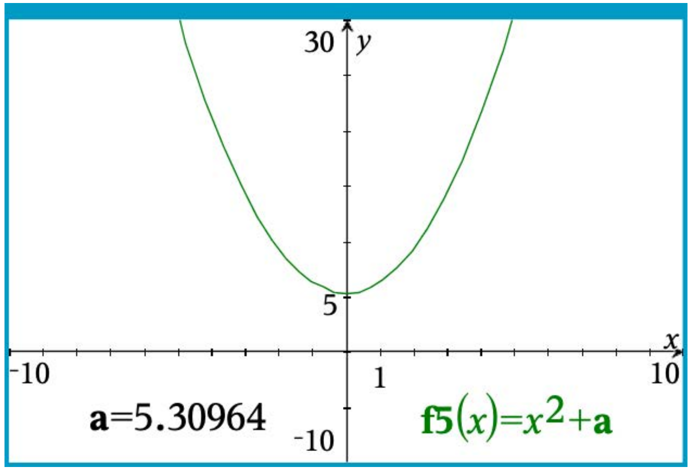
# Reële functies wat anders bekeken

## *Functies van de $2^e$ graad*



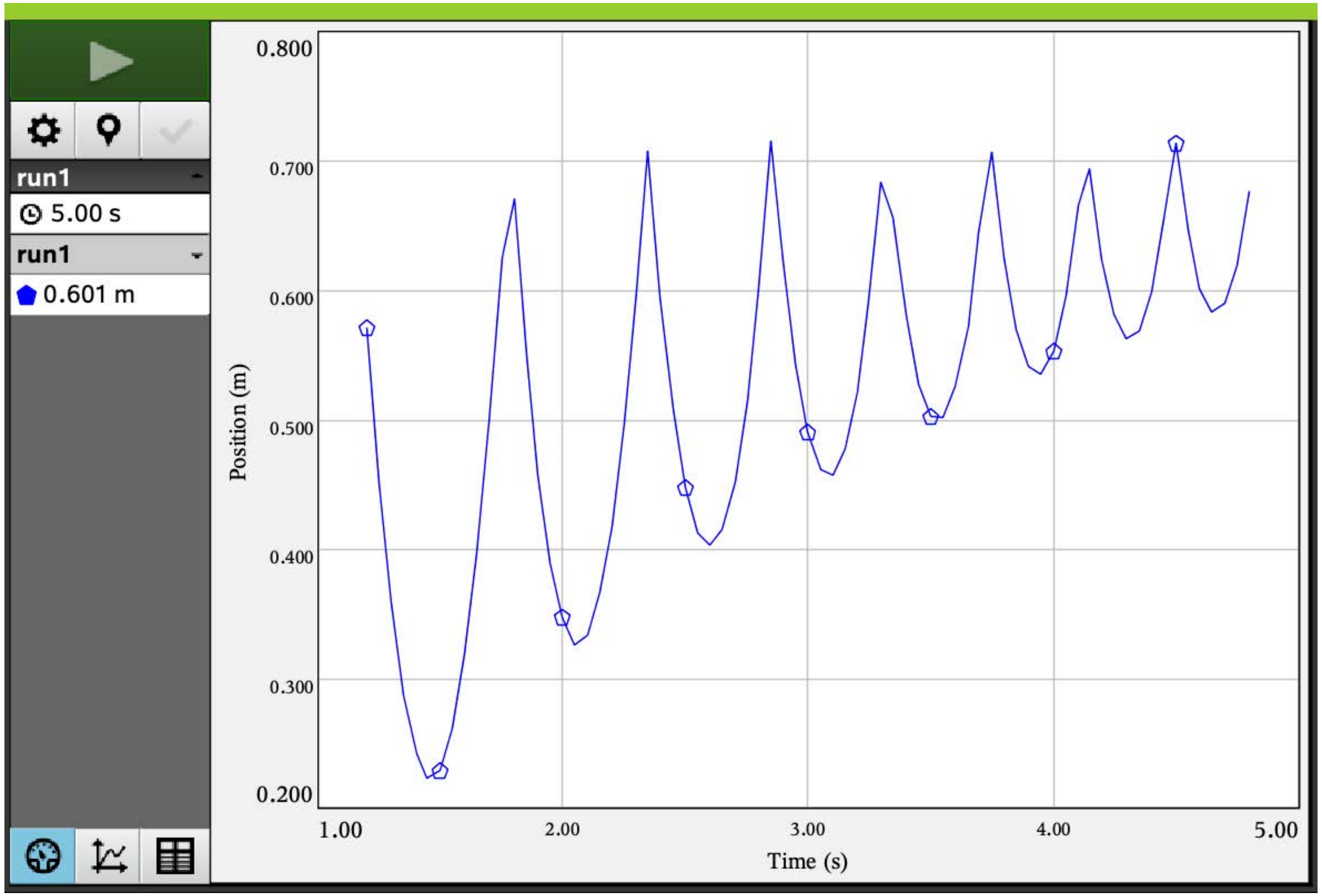



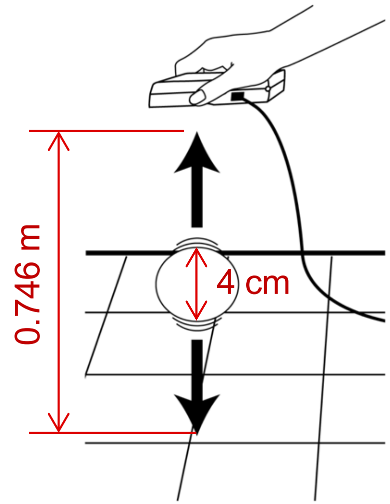



# Reële functies wat anders bekeken

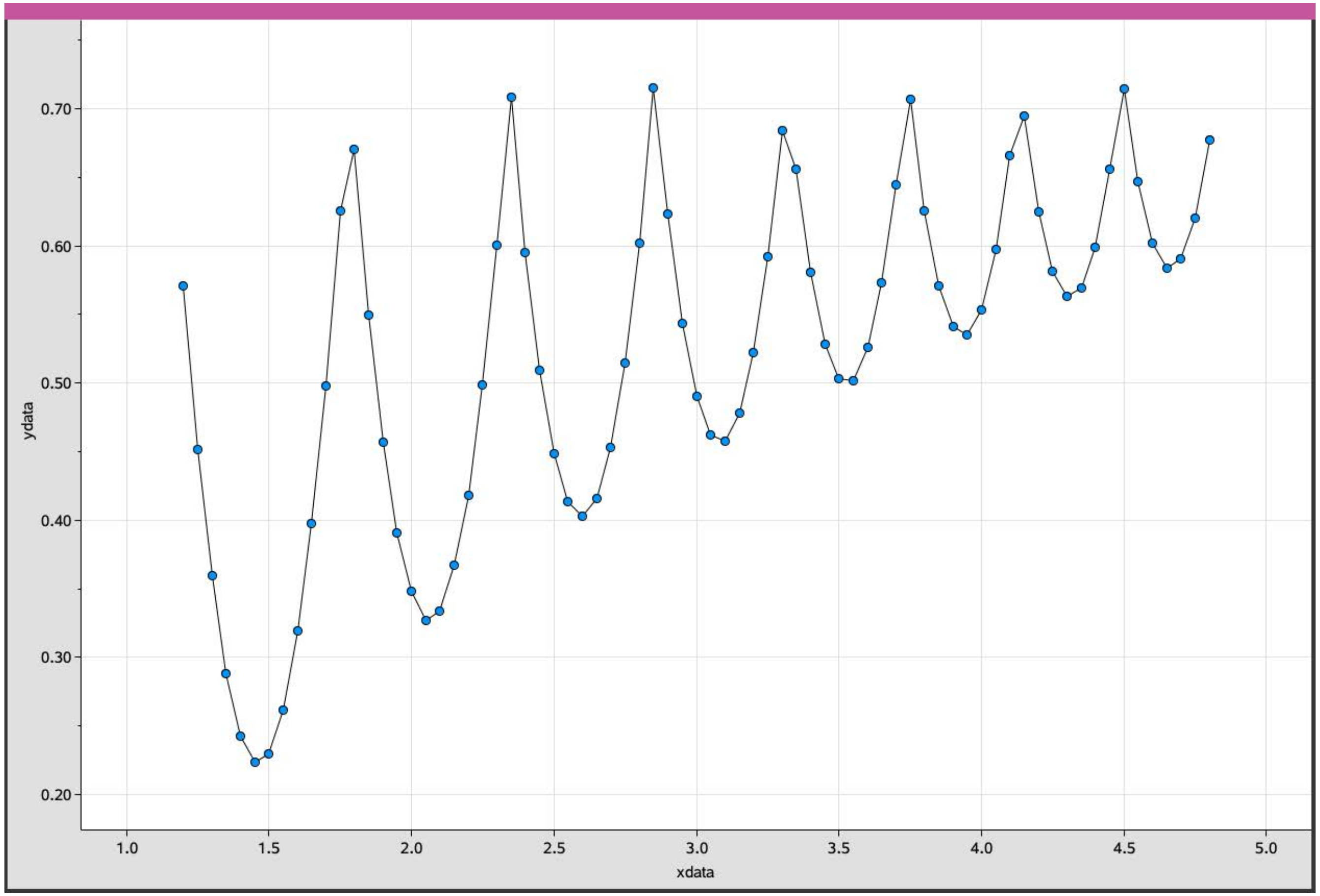
## *Bouncing Ball*



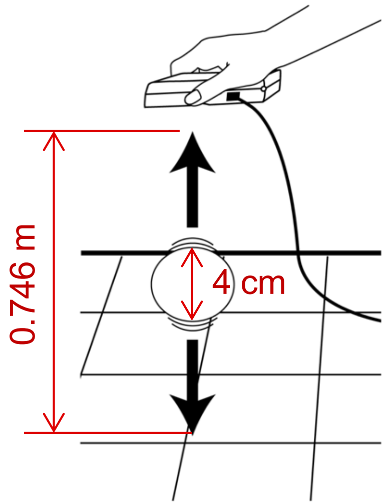

# Reële functies wat anders bekeken

## *Bouncing Ball*

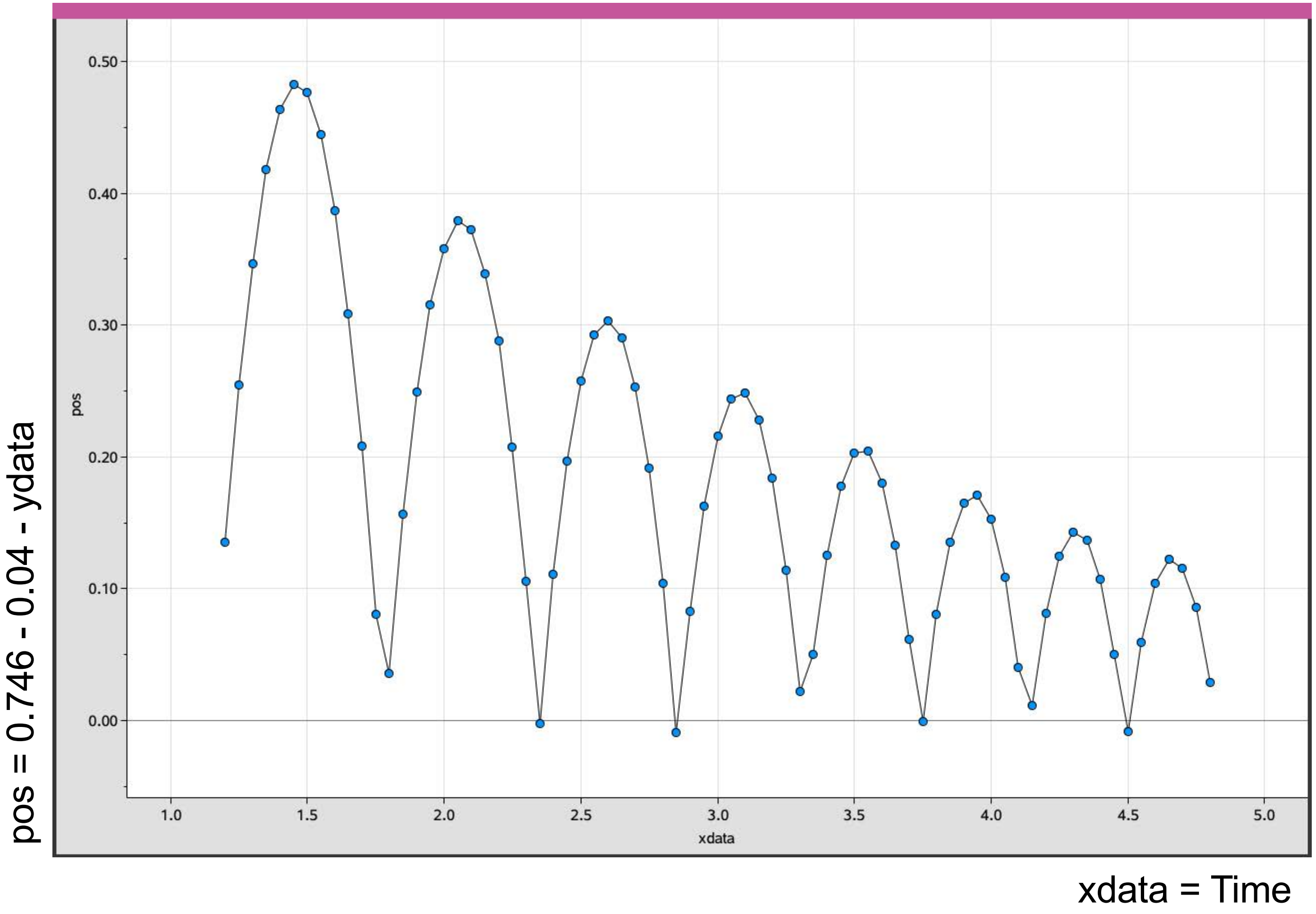


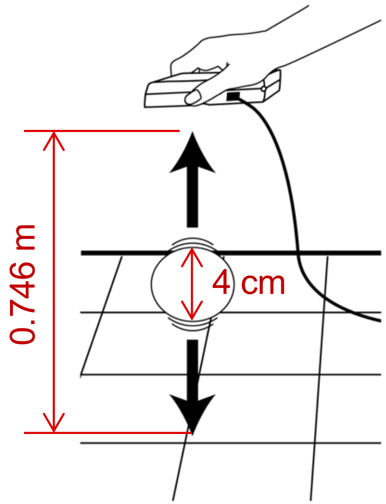

ydata = Position

xdata = Time




# Reële functies wat anders bekeken
## *Bouncing Ball*

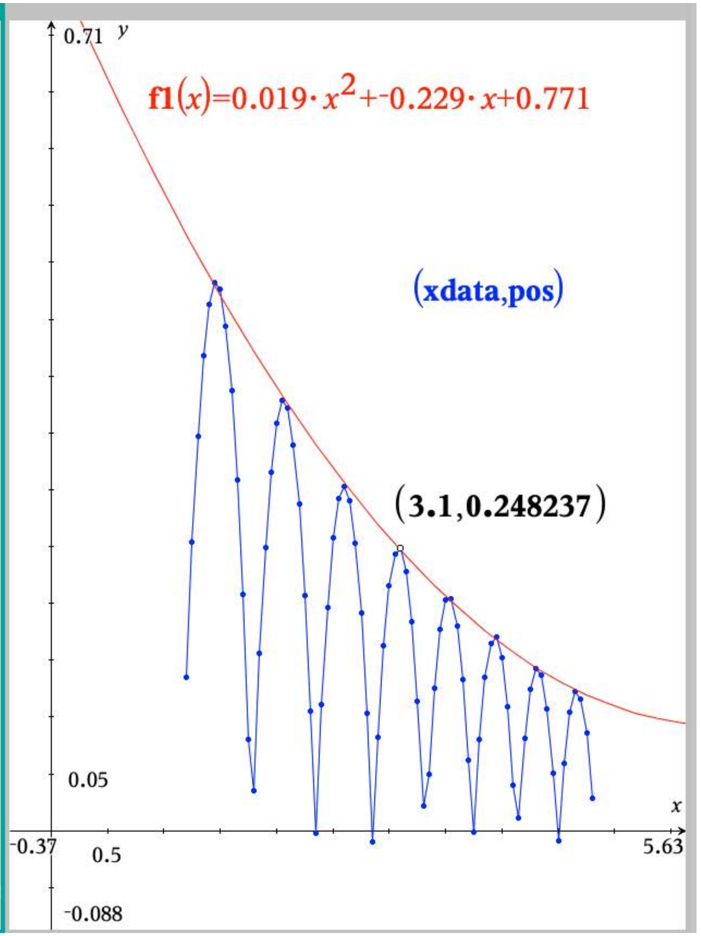


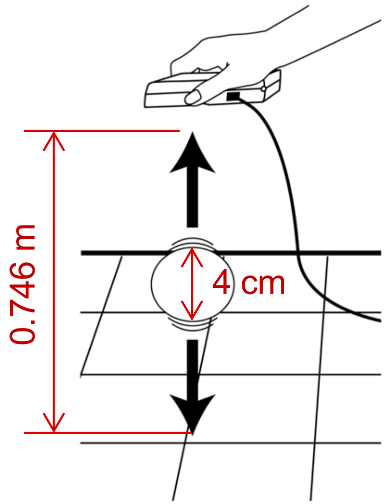




# Reële functies wat anders bekeken
## *Bouncing Ball*

| | A tijd | B afstand | C | D | E |
|---|---|---|---|---|---|
| = | =capture(: | =capture( | | | =QuadReg('tijd,'afstand, |
| 1 | 1.45 | 0.482649 | | Title | Quadratic Regression |
| 2 | 2.05 | 0.378812 | | RegEqn | a*x^2+b*x+c |
| 3 | 2.6 | 0.302904 | | a | 0.019346 |
| 4 | 3.1 | 0.248237 | | b | -0.229055 |
| 5 | 3.55 | 0.204286 | | c | 0.770959 |
| 6 | 3.95 | 0.170739 | | R² | 0.999479 |
| 7 | 4.3 | 0.143017 | | Resid | {0.0031449786614919... |
| 8 | 4.65 | 0.122121 | | | |

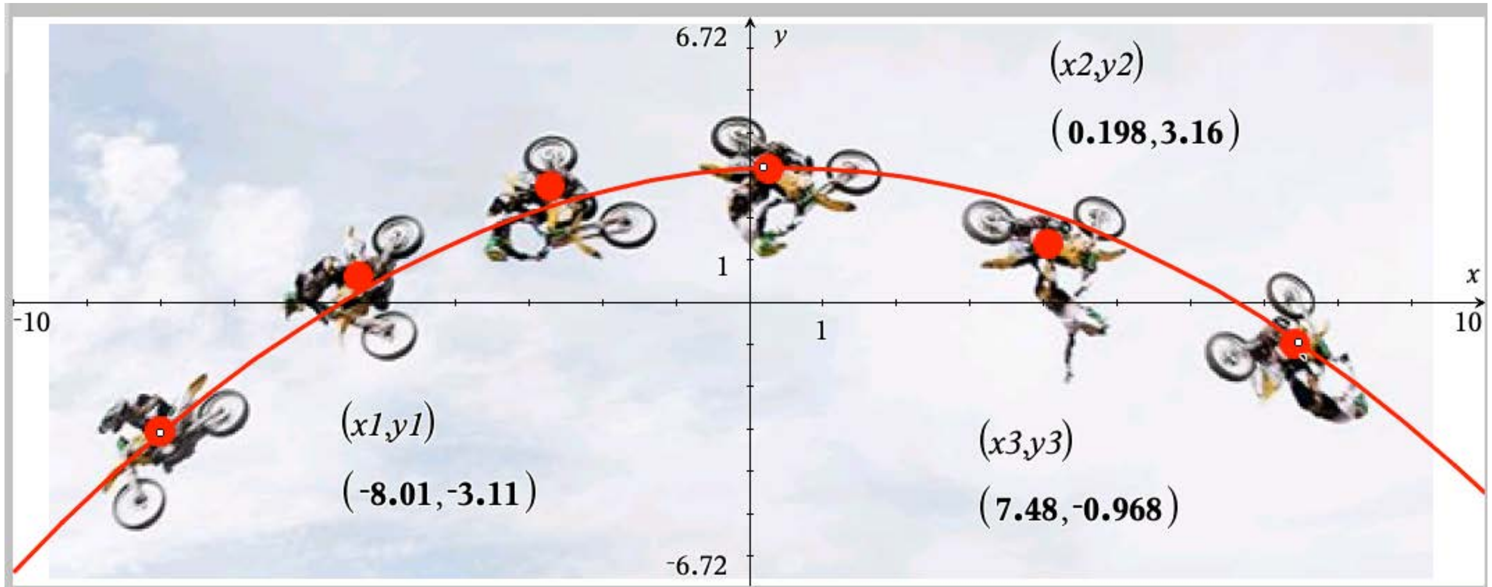
*E1* ="Quadratic Regression"




# Reeële functies wat anders bekeken

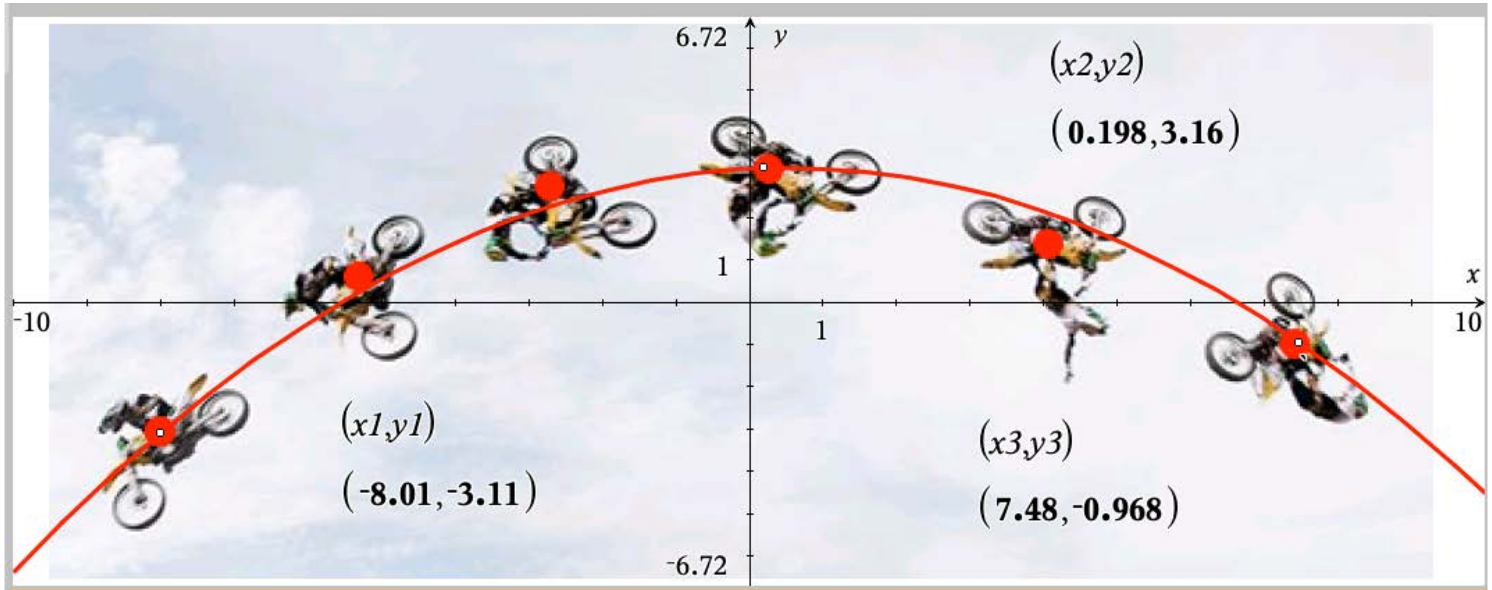
*Quadratic Modelling*




$$\text{coeff}:=\text{zeros}\left(\begin{cases} a\cdot \mathbf{x1}^2+b\cdot \mathbf{x1}+c-\mathbf{y1} \\ a\cdot \mathbf{x2}^2+b\cdot \mathbf{x2}+c-\mathbf{y2} \\ a\cdot \mathbf{x3}^2+b\cdot \mathbf{x3}+c-\mathbf{y3} \end{cases},\{a,b,c\}\right) \blacktriangleright \begin{bmatrix} -0.086 & 0.093 & 3.15 \end{bmatrix}$$

$$\text{Define } \mathbf{f1}(x)=\mathbf{coeff}[1,1]\cdot x^2+\mathbf{coeff}[1,2]\cdot x+\mathbf{coeff}[1,3]=-0.086\cdot x^2+0.093\cdot x+3.15$$

# Reële functies wat anders bekeken
## *Quadratic Modelling*



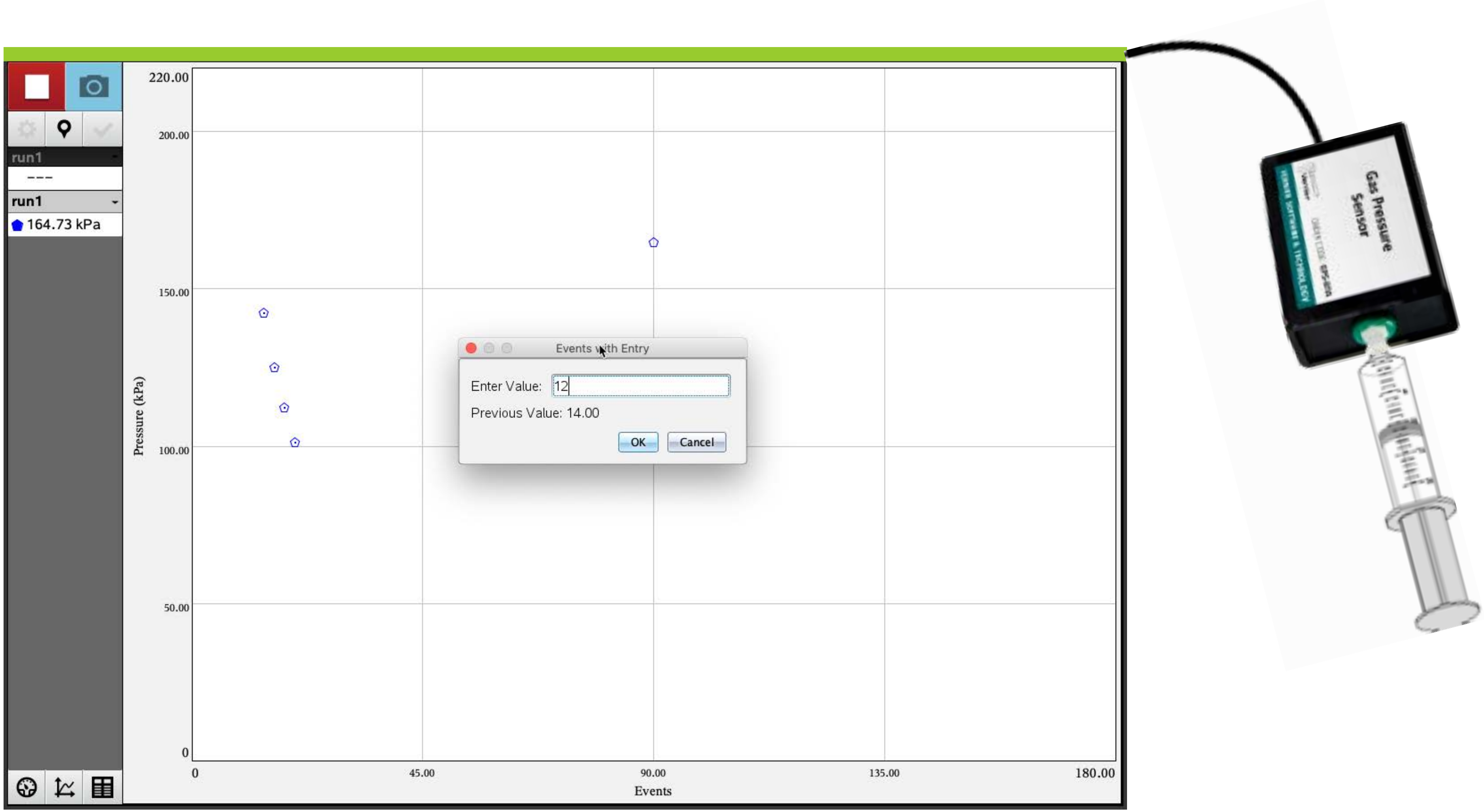

$a$

$$\begin{bmatrix} 64.0939 & -8.00587 & 1 & 3.1067 \\ 0.039323 & 0.1983 & 1 & -3.16166 \\ 55.2278 & 7.43154 & 1 & 1.00265 \end{bmatrix}$$

$\text{rref}(a)$

$$\begin{bmatrix} 1. & 0. & 0. & 0.086787 \\ 0. & 1. & 0. & -0.086451 \\ 0. & 0. & 1 & -3.14793 \end{bmatrix}$$

# Reële functies wat anders bekeken
## *Boyle's Law*



# Reële functies wat anders bekeken

*Boyle's Law*

| | A volume | B druk | C prod | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| = | ='run1.events | ='run1.pressure | =volume*druk | | | | | | |
| 1 | 20. | 101.309 | 2026.18 | | mean | | | | |
| 2 | 18. | 112.391 | 2023.04 | | 1995.92 | | | | |
| 3 | 16. | 125.072 | 2001.15 | | | | | | |
| 4 | 14. | 142.352 | 1992.93 | | | | | | |
| 5 | 12. | 164.732 | 1976.78 | | | | | | |
| 6 | 10. | 195.541 | 1955.41 | | | | | | |

E1 " mean"